Chapter 1

Preordered Sets, Posets and Lattices

This chapter introduces the most basic constructs of order theory. In the decreasing order of generality, these are the notions of preorders, partial orders, lattice orders, and finally, linear orders. In particular, we discuss some basic features of these notions of “ordering,” and look at several examples of them. We also show how these notions arise naturally in individual decision theory, and hence in economics. In the latter part of the chapter we use our ordering concepts to introduce a generalization of the notion of “monotonic function” and then extend this further to the context of set-valued maps. A brief introduction to the theory of Galois connections is also outlined.

All results that we shall derive in this section are elementary, but they will later act as the building blocks of some deeper results of order theory.

1. Binary Relations

Let $X$ be a nonempty set. A nonempty subset $R$ of $X \times X$ is called a binary relation on $X$. If $(x,y) \in R$, then we think of $R$ as associating the object $x$ with $y$, and if $\{(x,y),(y,x)\} \cap R = \emptyset$, we understand that there is no connection between $x$ and $y$ as instigated by $R$. In what follows, we adopt the convention of writing

$$x R y$$

instead of $(x,y) \in R$. Moreover, we simply write $x R y R z$ to mean $x R y$ and $y R z$, and so on.

The following definition catalogues a few of the most basic properties that a binary relation may possess.

Definition. A binary relation $R$ on a nonempty set $X$ is said to be reflexive if

$$x R x \quad \text{for every } x \in X.$$
It is **complete** if
\[ \text{either } x \, R \, y \text{ or } y \, R \, x \text{ for every } x, y \in X, \]

**symmetric** if
\[ x \, R \, y \text{ implies } y \, R \, x \text{ for every } x, y \in X, \]
and **antisymmetric** if
\[ x \, R \, y \, R \, x \text{ implies } x = y \text{ for every } x, y \in X. \]

Finally, we say that \( R \) is **transitive** if
\[ x \, R \, y \, R \, z \text{ implies } x \, R \, z \text{ for every } x, y, z \in X. \]

Any binary relation can be decomposed into two binary relations, one symmetric (the weak part of the relation) and one asymmetric (its strong part).

**Definition.** Let \( R \) be a binary relation on a nonempty set \( X \). The **asymmetric part** of \( R \) is defined as the binary relation \( P_R \) on \( X \) with
\[ x \, P_R \, y \iff x \, R \, y \text{ but not } y \, R \, x. \]

The binary relation \( I_R := R \setminus P_R \) on \( X \) is then called the **symmetric part** of \( R \).

For any binary relation \( R \) on a nonempty set \( X \), the binary relations \( P_R \) and \( I_R \) are disjoint, and we have \( R = P_R \cup I_R \). Here \( I_R \) is a symmetric binary relation on \( X \), which is reflexive if so is \( R \). By contrast, \( P_R \) is neither reflexive nor symmetric. Finally, if \( R \) is transitive, so are \( P_R \) and \( I_R \). (The proofs of these statements are easy, and hence omitted.)

**Note.** For any elements \( x, y \) and \( z \) in \( X \), and any binary relation \( R \) on \( X \),
\[ x \, P_R \, y \, R \, z \text{ (or } x \, R \, y \, P_R \, z) \text{ implies } x \, P_R \, z. \]

**Exercises**

1.1. Let \( \triangleright \) be a binary relation on a nonempty set \( X \) such that, for any \( x, y, z \in X \),
(a) [**Asymmetry**] \( x \triangleright y \) implies that \( y \triangleright x \) is false; and
(b) [**Negative Transitivity**] \( x \triangleright z \) implies either \( x \triangleright y \) or \( y \triangleright z \).
Define the binary relation \( \trianglerighteq \) on \( X \) by \( x \trianglerighteq y \) iff \( y \triangleright x \) is false. Show that \( \trianglerighteq \) is a complete preorder on \( X \) whose asymmetric part equals \( \triangleright \).
1.2. Let $R$ be a reflexive binary relation on a nonempty set $X$, and let $\mathcal{R}$ be the collection of all preorders on $X$ that contain $R$. For each positive integer $m$, define the relation $R_m$ on $X$ by $x R_m y$ iff there exist $z_1, ..., z_m \in X$ such that $x R z_1 R \cdots R z_m R y$. In turn, the binary relation

$$\text{tran}(R) := R \cup R_1 \cup R_2 \cup \cdots$$

on $X$ is called the transitive closure of $R$. Show that $\text{tran}(R)$ belongs to $\mathcal{R}$ and we have $\text{tran}(R) \subseteq R'$ for every $R'$ in $\mathcal{R}$.

1.3. Let $\succsim$ be a binary relation on a nonempty set $X$. We say that $\succsim$ is acyclic if there does not exist a positive integer $k$ and elements $x_1, ..., x_k$ in $X$ such that $x_1 \succsim \cdots \succsim x_k \succsim x_1$. Prove that transitivity of $\succsim$ implies its acyclicity. Give an example to show that the converse is false.

1.4. Let $\succsim$ be a binary relation on a nonempty set $X$. Prove that $\succsim$ is acyclic iff for every nonempty finite subset $S$ of $X$, there is an $x \in S$ such that $\omega \succsim x$ does not hold for any $\omega \in S$.

2. Equivalence Relations

The binary relations that are reflexive, symmetric and transitive are encountered in mathematical analysis routinely. They are given a special name.

**Definition.** A binary relation $\sim$ on a nonempty set $X$ is called an equivalence relation if it is reflexive, symmetric and transitive. For any $x \in X$, the equivalence class of $x$ relative to $\sim$ is defined as the set

$$[x]_\sim := \{\omega \in X : x \sim \omega\}.$$

The collection of all equivalence classes relative to $\sim$ – denoted as $X/\sim$ – is called the quotient set of $X$ relative to $\sim$, that is,

$$X/\sim := \{[x]_\sim : x \in X\}.$$

An equivalence relation can be used to decompose a grand set of interest into subsets such that the members of the same subset are thought of as “identical” while the members of distinct subsets are viewed as “distinct.” To say this formally, let us recall that a partition of a nonempty set $X$ is a collection $\mathcal{A}$ of nonempty subsets of $X$ such that $\bigcup \mathcal{A} = X$ and $A \cap B = \emptyset$ for every distinct $A$ and $B$ in $\mathcal{A}$. Not surprisingly, the class of equivalence classes induced by any equivalence relation on a set is a partition of that set.

**Proposition 2.1.** For any equivalence relation $\sim$ on a nonempty set $X$, the quotient set $X/\sim$ is a partition of $X$. 

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We leave the proof of this fact as an (easy) exercise.

**Example 2.1.** For any nonempty set $X$, the **diagonal relation**

$$\Delta_X := \{(x, x) : x \in X\}$$

is the smallest equivalence relation that can be defined on $X$ (in the sense that $\Delta_X$ is contained in every equivalence relation on $X$). Clearly, $[x]_{\Delta_X} = \{x\}$ for every $x \in X$, which shows that $\Delta_X$ is none other than the “equality” relation on $X$. We have $X/\Delta_X = \{\{x\} : x \in X\}$. At the other extreme is $X \times X$, which is the largest equivalence relation that can be defined on $X$. We have $[x]_{X \times X} = X$ for every $x$ in $X$, and hence this relation induces the coarsest possible partition of $X$, that is, $X/X_{X \times X} = \{X\}$.

**Example 2.2.** The symmetric part of any reflexive and transitive binary relation on a nonempty set is an equivalence relation on that set.

**Exercises**

2.1. Prove Proposition 2.1.

2.2. **(Converse of Proposition 2.1)** Let $\mathcal{A}$ be a partition of a nonempty set $X$, and define the binary relation $\sim$ on $X$ by $x \sim y$ iff $\{x, y\} \subseteq A$ for some $A \in \mathcal{A}$. Show that $\sim$ is an equivalence relation on $X$.

2.3. **(Factorization of a Function)** Let $X$ and $Y$ be two nonempty sets and $f : X \rightarrow Y$ a function. Define the equivalence relation $\sim$ on $X$ by $x \sim y$ iff $f(x) = f(y)$. Prove that there is a surjection $g : X \rightarrow X/\sim$ and an injection $h : X/\sim \rightarrow Y$ such that $f = h \circ g$. (Thus: Every function is a composition of a surjection and an injection.)

**3. Order Relations**

**3.1 Definitions and Examples**

The following are some of the most fundamental concepts of order theory.

**Definition.** A binary relation $\preceq$ on a nonempty set $X$ is said to be a **preorder** on $X$ if it is transitive and reflexive. It is said to be a **partial order** on $X$ if it is an antisymmetric preorder on $X$. Finally, a complete partial order on $X$ is said to be a **linear order** (or a **chain**) on $X$. 

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**Notation.** For any preorder \( \succeq \), we denote by \( \succ \) the asymmetric part of \( \succeq \), and by \( \sim \) the symmetric part of \( \succeq \).

A preorder on a nonempty set \( X \) may have a large symmetric part (which is necessarily an equivalence relation on \( X \)). By contrast, the symmetric part of a partial order on \( X \) is the smallest reflexive relation on \( X \), that is, it equals \( \Delta_X \). This is the main difference between a preorder and a partial order.

**Definition.** A preordered set is an ordered pair \( (X, \succeq) \), where \( X \) is a nonempty set and \( \succeq \) is a preorder on \( X \). We say that \( (X, \succeq) \) is a finite preordered set, if \( (X, \succeq) \) is a preordered set and \( |X| < \infty \).

We shall make use of the following bit of notation quite frequently when working with preordered sets.

**Notation.** Given a preordered set \( (X, \succeq) \) and an element \( x \) of \( X \), we denote by \( x^\downarrow \) the set of all elements of \( X \) that are ranked lower than \( x \) by \( \succeq \), that is,

\[
x^\downarrow := \{ \omega \in X : x \succeq \omega \}.
\]

Similarly,

\[
x^\uparrow := \{ \omega \in X : \omega \succeq x \}.
\]

**Note.** For any preordered set \( (X, \succeq) \) and \( x \in X \), we have \( x^\downarrow \cap x^\uparrow = [x]_\succeq \). In particular, if \( (X, \succeq) \) is a poset, then \( x^\downarrow \cap x^\uparrow = \{x\} \).

The following are two of the most important types of preordered sets.

**Definition.** A preordered set \( (X, \succ) \) is called a poset (short for partially ordered set) if \( \succ \) is antisymmetric, that is, if \( \succ \) is a partial order on \( X \). Similarly, \( (X, \succ) \) is called a loset (short for linearly ordered set) if \( \succ \) is antisymmetric and complete, that is, if \( \succ \) is a linear order on \( X \).

**Note.** A loset is also referred to as a chain in the literature. We shall occasionally use this terminology as well.
Note. If \((X, \succeq)\) is a preordered set, then so is \((X, \preceq)\). As we shall see, this duality is exploited routinely in order theory.

**Definition.** Let \((X, \preceq)\) be a preordered set. An **extension** of \(\preceq\) is a preorder \(\succeq\) on \(X\) such that \(\preceq \subseteq \succeq\) and \(\succ \subseteq \succeq\), where \(\succ\) and \(\succeq\) are the asymmetric parts of \(\preceq\) and \(\succeq\), respectively.

Intuitively speaking, an extension of the preorder \(\preceq\) on a nonempty set \(X\) is “more complete” than \(\preceq\) in the sense that it compares more elements, but it certainly agrees with \(\preceq\) when the latter applies. If \(\succeq\) is a partial order, then it is an extension of \(\preceq\) iff \(\preceq \subseteq \succeq\). (Why?)

We now present several examples of posets. We will encounter many other examples throughout these lecture notes.

**Example 3.1.** Let \(X\) be a nonempty set. Then, \((X, \Delta_X)\) is a poset, and for any \(x\) in \(X\), we have \(x^\perp = \{x\} = x^\uparrow\) in the context of this poset. In fact, the diagonal relation \(\Delta_X\) is the only partial order on \(X\) which is also an equivalence relation. Moreover, every reflexive relation on \(X\) is an extension of \(\Delta_X\).

The binary relation \(X \times X\) is, on the other hand, a complete preorder. This preorder is not a partial order unless \(|X| = 1\). (For any \(x\) in \(X\), we have \(x^\perp = X = x^\uparrow\) in the context of the poset \((X, X \times X)\).) Furthermore, the only extension of \(X \times X\) is itself.

**Example 3.2.** \((\mathbb{R}, \geq)\) is a loset, where \(\geq\) is the usual linear order on \(\mathbb{R}\). In the context of this loset, we have \(x^\perp = [x, \infty)\) and \(x^\uparrow = (-\infty, x]\) for any real number \(x\).

**Example 3.3.** \((\mathbb{R}^n, \geq)\) is a poset for any positive integer \(n\), where \(\geq\) is defined coordinatewise, that is,

\[
\mathbf{x} \succeq \mathbf{y} \quad \text{iff} \quad x_i \geq y_i \text{ for each } i = 1, \ldots, n.
\]

When we talk of \(\mathbb{R}^n\) without specifying an alternative preorder, we have this partial order in mind. (Note. Throughout this text, we use the same notation for the (partial) order of \(\mathbb{R}^n\) and (linear) order of \(\mathbb{R}\).)

For each \(k \in \{1, \ldots, n\}\), define the relation \(\succeq^k\) on \(\mathbb{R}^n\) by

\[
\mathbf{x} \succeq^k \mathbf{y} \quad \text{iff} \quad x_1 + \cdots + x_k \geq y_1 + \cdots + y_k.
\]

Then, \(\succeq^k\) is a preorder on \(\mathbb{R}^n\); it is an extension of \(\geq\) iff \(k = n\).
Example 3.4. Let $X$ be a nonempty set. Then, $(2^X, \supseteq)$ is a poset, and for any $A \subseteq X$, we have $A^\dagger = 2^\dagger$ in the context of this poset. The linear order $\supseteq$ on $2^X$, defined by

$$A \supseteq B \iff |A| \geq |B|,$$

is an extension of $\supseteq$, provided that $X$ is a finite set.

Example 3.5. Let $C[0,1]$ stand for the linear space of continuous real maps on the interval $[0,1]$. Then, $(C[0,1], \geq)$ is a poset, where

$$f \geq g \iff f(t) \geq g(t) \text{ for every } t \text{ in } [0,1].$$

The linear order $\supseteq$ on $C[0,1]$, defined by

$$f \supseteq g \iff \int_0^1 f(t)dt \geq \int_0^1 g(t)dt,$$

is an extension of $\supseteq$.

Example 3.6. Let $X$ be a linear space and $C$ a convex cone in $X$ (that is, $C$ is a subset of $X$ such that $\lambda C + C \subseteq C$ for every real number $\lambda \geq 0$). Define the binary relation $\geq_C$ on $X$ by

$$x \geq_C y \iff x \in y + C.$$

Then, $(X, \geq_C)$ is a preordered set. (We say that $\geq_C$ is a vector preorder on $X$, and that $(X, \geq_C)$ is a preordered linear space.) In the context of this preordered set, we have

$$x^\dagger = x - C \quad \text{and} \quad x^\ddagger = x + C$$

for any $x$ in $X$.

It is worth noting that a vector preorder $\geq_C$ is compatible with the operations of addition and scalar multiplication in the sense that $x \geq_C y$ iff $\lambda x + z \geq_C \lambda y + z$ for every $x, y, z \in X$ and $\lambda > 0$. Furthermore, this preorder is a partial order iff $C \cap -C$ contains only the origin of $X$. Under this condition, another vector partial order $\supseteq_D$ on $X$ is an extension of $\supseteq_C$ iff $C \subseteq D$.

We conclude with one final definition.

Definition. The product of two preordered sets $(X, \preceq_X)$ and $(Y, \preceq_Y)$ is the preordered set $(X \times Y, \preceq)$, where

$$(x, y) \preceq (z, w) \iff x \preceq_X z \text{ and } y \preceq_Y w.$$
This definition is extended to the case of products of an arbitrary (finite) number of preordered sets by mathematical induction.

**Example 3.7.** For any positive integer \( n \geq 2 \), the poset \( (\mathbb{R}^n, \succeq) \) is the product of \( n \) copies of the loset \( (\mathbb{R}, \geq) \). Equivalently, it is the product of \( (\mathbb{R}^k, \geq) \) and \( (\mathbb{R}^{n-k}, \geq) \) for any \( k \in \{1, \ldots, n-1\} \).

### 3.2 Preorders in Decision Theory

In passing, let us note that preordered sets arise naturally in the context of decision theory. Indeed, in individual choice theory, a **preference relation** \( \succsim \) on a nonempty alternative set \( X \) is defined merely as a preorder on \( X \). We think of \( X \) as the universal space of choice items in this context. In turn, \( \succsim \) is assumed to contain all the information that concerns one’s comparative preferences about the outcomes in \( X \). If \( x \succsim y \) holds, we understand that the individual with preference relation \( \succsim \) views the alternative \( x \) at least as good as the alternative \( y \). Thus, in the context of \( (X, \succsim) \), the set of all alternatives in \( X \) that are at least as good as \( x \) for the involved individual corresponds to \( x\uparrow \), and \( x\downarrow \) is similarly interpreted.

Induced from \( \succsim \) are the **strict preference relation** \( \succ \) on \( X \), which is the asymmetric part of \( \succsim \), and the **indifference relation** \( \sim \) on \( X \), which is the symmetric part of \( \succsim \). When \( x \succ y \) holds, we understand that the agent is strictly better off with the alternative \( x \) relative to \( y \). (We may think of this as saying that the agent would be willing to pay a positive price to be able to move from \( y \) to \( x \).) When \( x \sim y \) is the case, we think of the agent being indifferent between the alternatives \( x \) and \( y \). (The agent would then not pay a positive price to switch from \( x \) to \( y \), and vice versa.) Finally, if neither \( x \succsim y \) nor \( y \succsim x \), a situation which is denoted as

\[
x \not\succeq y,
\]

we understand that the agent is **indecisive** about comparing \( x \) to \( y \).

**Remark.** Given a preference relation on a nonempty alternative set \( X \), and any alternative \( x \) in \( X \), the equivalence class \([x]_\sim \) is called the **indifference class** of \( x \) in the context of choice theory. This is a straightforward generalization of the familiar concept of “the indifference curve that passes through \( x \)” In particular, Proposition 2.1 says that no two distinct indifference sets can have a point in common. (In intermediate microeconomics, this is often paraphrased as: “Indifference curves do not cross.”)
Exercises

3.1. Let $X$ be a topological space, and define the binary relation $\preceq$ on $X$ by $x \preceq y$ iff every open neighborhood of $y$ is also an open neighborhood of $x$. Prove that $\preceq$ is a preorder on $X$, but it need not be a partial order. Also show that if $X$ is Hausdorff, then $\preceq$ is a partial order on $X$.

3.2. Let $(X, \succeq)$ be a preordered set. For any nonempty subset $S$ of $X$, we define

$$S^\downarrow := \bigcup\{x^\downarrow : x \in S\} \quad \text{and} \quad S^\uparrow = \bigcup\{x^\uparrow : x \in S\}.$$ 

Prove: For any nonempty $S \subseteq X$,

$$S^\downarrow = S^\downarrow \quad \text{and} \quad S^\uparrow = S^\uparrow,$$

while

$$S \subseteq S^\downarrow \cap S^\uparrow.$$

Give an example to show that equality need not hold in this statement.

3.3. Let $X$ be a nonempty set and $S \subseteq 2^X$. In the context of $(2^X, \supseteq)$, compute $S^\downarrow \cap S^\uparrow$.

3.4. In the context of $(\mathbb{R}^n, \succcurlyeq)$, compute $0^\downarrow \cap 0^\uparrow$.

3.5. Give an example of a poset $(X, \succcurlyeq)$ such that there is an $S \subseteq X$ with $S \subseteq S^\downarrow \subseteq S^\downarrow \subseteq \cdots$

3.6. Let $(X, \succeq)$ be a preordered set, and $D$ a nonempty subset of $X$. We say that $D$ is $\succeq$-directed if for every $x, y \in D$, there is a $z \in D$ with $z \preceq x$ and $z \preceq y$. Show that $D$ is $\preceq$-directed iff so is $D^\downarrow$.

3.7. (Direct Sum of Posets) Let $(X, \succcurlyeq_X)$ and $(Y, \succcurlyeq_Y)$ be two posets. We define the binary relation $\succcurlyeq_{X \oplus Y}$ on $X \cup Y$ by

$$x \succcurlyeq_{X \oplus Y} y \quad \text{iff} \quad (x, y) \in \succcurlyeq_X \cup \succcurlyeq_Y \cup (X \times Y).$$

Show that $(X \cup Y, \succcurlyeq_{X \oplus Y})$ is a poset. (This poset is called the direct sum of $(X, \succcurlyeq_X)$ and $(Y, \succcurlyeq_Y)$.)

3.8. (Completion of a Preorder) A complete preorder that extends a preorder $\succeq$ is said to be a completion of $\succeq$. Prove that every preorder on a nonempty finite set admits a completion.

3.9. (The Scott-Suppes Theorem) Let $I$ be the collection of all closed intervals of unit length, and define the binary relation $\succeq$ on $I$ by $[a, b] \succeq [c, d]$ iff either $a \geq d$ or $[a, b] = [c, d]$. We say that a binary relation $\succeq$ on a nonempty set $X$ is a semiorder on $X$ if there is a bijection $\varphi : X \to I$ such that $x \succeq y$ iff $\varphi(x) \succeq \varphi(y)$.

Let $\succeq$ be a semiorder on $X$. Prove first that $(X, \succeq)$ is a poset. Next, assume that $X$ is finite, and prove that there exists a function $f : X \to \mathbb{R}$ such that

$$x \succeq y \quad \text{iff} \quad f(x) > f(y) + 1$$

for every $x$ and $y$ in $X$. 

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4. Representation Through Complete Preorders

4.1 Representation of a Complete Binary Relation

There is an interesting way of representing a complete binary relation on a nonempty finite set $X$ in terms of complete preorders on $X$.

**Theorem 4.1.1.** Let $X$ be a nonempty finite set and $R$ a complete binary relation on $X$. Then, there exists a nonempty set $\mathcal{P}$ of complete preorders on $X$ such that

$$ x R y \iff |\{\succsim \in \mathcal{P} : x \succsim y\}| \geq |\{\succsim \in \mathcal{P} : y \succsim x\}| $$

for every $x$ and $y$ in $X$.

This result is due to McGarvey (1953), whose main interest was to understand the structure of social preferences that arise from the majority voting of finitely many individuals with complete preference relations. From this viewpoint, Theorem 4.1.1 says that, when the outcome space is finite, any complete binary relation can be rationalized this way. Or, put differently, every complete binary relation $R$ on a finite set $X$ can be interpreted “as if” it arises from the majority voting of a group of agents with complete preference relations on $X$.

**Proof of Theorem 4.1.1.** If $|X| \leq 2$, the claim is trivial, so we assume $n := |X| > 2$ in what follows. Take any two elements $x$ and $y$ in $X$ with $x R y$, and let $\{a_1, \ldots, a_{n-2}\}$ be an enumeration of $X \setminus \{x, y\}$. If $x P_R y$, then we define $\succsim_{xy}$ as the partial order on $X$ with

$$ x \succsim_{xy} y \succsim_{xy} a_1 \succsim_{xy} \cdots \succsim_{xy} a_{n-2}, $$

and $\succeq_{xy}$ as the partial order on $X$ with

$$ a_{n-2} \succeq_{xy} \cdots \succeq_{xy} a_1 \succeq_{xy} x \succeq_{xy} y. $$

(Here, of course, $\succsim_{xy}$ is the asymmetric part of $\succsim_{xy}$, and similarly for $\succeq_{xy}$.) On the other hand, if $x I_R y$, we leave $\succsim_{xy}$ exactly as above, and define $\succeq_{xy}$ as the partial order on $X$ with

$$ a_{n-2} \succeq_{xy} \cdots \succeq_{xy} a_1 \succeq_{xy} y \succeq_{xy} x. $$

Finally, we define

$$ \mathcal{P} := \{\succsim_{xy} : (x, y) \in X^2 \setminus \Delta_X\} \cup \{\succeq_{xy} : (x, y) \in X^2 \setminus \Delta_X\}. $$

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1For a much deeper result along these lines, see Kalai (2006).
Observe that, by construction, \( x \mathrel{P} R \ y \) implies
\[
|\left\{ z \in P : x \succ y \right\} | = 2 \geq 0 = |\left\{ z \in P : y \succ x \right\} |
\]
and \( x \mathrel{I} R \ y \) implies
\[
|\left\{ z \in P : x \succ y \right\} | = 0 = |\left\{ z \in P : y \succ x \right\} |,
\]
and hence the proof.\(^2\)

**Remark.** There is a literature (that goes under the garb of *representation of finite tournaments*) that attempts to find \( P \) in Theorem 4.1.1 with the smallest cardinality. For instance, one can show that we can choose \( P \) in Theorem 4.1 with \( |P| \leq |X| + 2 \) when \( |X| \) is even, and with \( |P| \leq |X| + 1 \) when \( |X| \) is odd.\(^3\) (See Stearns, 1959). Better bounds than this are known, but these sorts of theorems have limited interest for decision theory. As decision theorists are more interested in the conceptual, as opposed to the combinatoric, meaning of Theorem 4.1.1, more interesting would be to extend this result to the case where the requirement of finiteness of \( X \) (but not of \( P \)) is relaxed. Next to nothing is known about this matter, however.

### 4.2 Representation of a Preorder

We can also represent a preorder on *any* nonempty set \( X \) in terms of complete preorders on \( X \).

**Proposition 4.2.1.** Let \( X \) be a nonempty set and \( \succeq \) a preorder on \( X \). Then, there exists a nonempty set \( P \) of complete preorders on \( X \) such that

\[
x \succeq y \quad \text{iff} \quad x \trianglerighteq y \quad \text{for every} \quad \trianglerighteq \in P
\]

for every \( x \) and \( y \) in \( X \).

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\(^2\)The idea here is that, when \( x \mathrel{P} R \ y \), the element \( x \) is ranked strictly higher than \( y \) by both \( \succ_{xy} \) and \( \succeq_{xy} \) (and hence both of these individuals “vote” for \( x \) over \( y \)). On the other hand, for any \((a, b)\) in \( X \times X \), distinct from \((x, y)\), the complete preorders \( \succ_{xy} \) and \( \succeq_{xy} \) disagree on the ranking of \( x \) and \( y \) (and hence the “votes” of the involved individuals cancel each other). The situation \( x \mathrel{I} R \ y \) is similarly analyzed.

\(^3\)In the proof above, we have included in \( P \) two preference relations for each pair of distinct elements in \( X \). Therefore, in our construction, \( |P| = 2 \binom{n}{2} = n(n - 1) \).
Proof. For any \( \omega \) in \( X \), we let \( f_\omega \) to stand for the indicator function of \( \omega \), that is, we define \( f_\omega : X \to \mathbb{R} \) by
\[
f_\omega(x) := \begin{cases} 
1, & \text{if } x \succsim \omega \\
0, & \text{otherwise}.
\end{cases}
\]
Next, for each \( \omega \) in \( X \), we define the complete preorder \( \succsim_\omega \) on \( X \) by
\[
x \succsim_\omega y \iff f_\omega(x) \geq f_\omega(y).
\]
It is an easy exercise to check that
\[
x \succsim y \iff x \succsim_\omega y \text{ for every } \omega \in X.
\]
Thus, letting \( \mathcal{P} := \{ \succsim_\omega : \omega \in X \} \) completes the proof.

Proposition 4.2.1 says that a preference relation \( \succsim \) of an individual on a nonempty set \( X \) can be interpreted “as if” this relation arises from the unanimity voting of a group of agents – these agents are usually thought of as the multiple “selves” (or “moods”) of the individual – with complete preference relations on \( X \). The situation \( x \succsim y \), that is, the case where the individual with preference relation \( \succsim \) is indecisive about how to compare the alternatives \( x \) and \( y \), is captured by the disagreement of the ranking of \( x \) and \( y \) by at least two “selves” of the individual.

Remark. In Proposition 4.2.1, no guarantee is given for the complete preorders in \( \mathcal{P} \) to be extensions of \( \succsim \). The representation certainly does not necessitate this to happen, and in fact, the construction we gave in the proof will fail to have this property in general. However, as we shall see later, it is actually possible to use only the extensions of \( \succsim \) in this representation in a variety of situations.

Exercises

4.1. Let \( \mathcal{P} \) be a nonempty collection of complete preorders on a nonempty set \( X \). Show that \( \bigcup \mathcal{P} \) is a complete and acyclic binary relation on \( X \).

4.2. Let \( \succsim \) be a complete and acyclic binary relation on a nonempty finite set \( X \). True or false: \( \succsim = \bigcup \mathcal{P} \) for some nonempty collection \( \mathcal{P} \) of complete preorders on \( X \).

4.3. Suggest and then establish a representation for acyclic binary relations on a nonempty finite set \( X \) in terms of complete preorders on \( X \).
5. Maxima and Minima

The extremal and extremum elements of a preordered set are essential to the analysis of that preordered set, and they arise routinely in applications. Put informally, an element of a preordered set is extremal if it is either never ranked strictly below another element in the set, or it is never ranked strictly above another element. By contrast, an extremum element is either ranked (weakly) higher than every other element in the set, or it is ranked (weakly) lower than every other element. Formally speaking:

**Definition.** Let \((X, \preceq)\) be a preordered set and \(S\) a nonempty subset of \(X\). An element \(x\) of \(S\) is called \(\preceq\)-maximal in \(S\) if there is no \(\omega \in S\) with \(\omega \succ x\). It is said to be \(\preceq\)-minimal in \(S\) if there is no \(\omega \in S\) with \(x \succ \omega\).

Such elements of a preordered set \((X, \preceq)\) are often called \(\preceq\)-extremal. In turn, the \(\preceq\)-extremum elements of this set are defined as follows:

**Definition.** Let \((X, \preceq)\) be a preordered set and \(S\) a nonempty subset of \(X\). An element \(x\) of a nonempty subset \(S\) of \(X\) is said to be a \(\preceq\)-maximum in \(S\) if \(x \preceq \omega\) for every \(\omega \in S\). It is said to be a \(\preceq\)-minimum in \(S\) if \(\omega \preceq x\) for every \(\omega \in S\).

**Notation.** Let \((X, \preceq)\) be a preordered set and \(S\) a nonempty subset of \(X\). We denote the set of all \(\preceq\)-maximal and \(\preceq\)-minimal elements in \(S\) by

\[
\text{MAX}(S, \preceq) \quad \text{and} \quad \text{MIN}(S, \preceq),
\]

respectively. Similarly, the set of all \(\preceq\)-maximum and \(\preceq\)-minimum elements in \(S\) are denoted by

\[
\max(S, \preceq) \quad \text{and} \quad \min(S, \preceq),
\]

respectively.

**Note.** For any preordered set \((X, \preceq)\), and any nonempty subset \(S\) of \(X\), we have

\[
\text{MAX}(S, \preceq) = \text{MIN}(S, \preceq) \quad \text{and} \quad \max(S, \preceq) = \min(S, \preceq).
\]

This duality allows one to deduce the properties minimal and minimum elements from those of maximal and maximum elements in general.
For any preordered set \((X, \preceq)\), and \(S \subseteq X\), an element of a subset \(S\) of \(X\) that is \(\preceq\)-maximum in \(S\) is \(\preceq\)-maximal in \(S\), that is, we always have
\[
\max(S, \preceq) \subseteq \text{MAX}(S, \preceq).
\]

Easy examples show that this containment can hold strictly. However, this happens only when \(S\) does not contain a \(\preceq\)-maximum element. That is,
\[
\max(S, \preceq) = \text{MAX}(S, \preceq) \quad \text{whenever} \quad \max(S, \preceq) \neq \emptyset.
\]

To see this, suppose \(\max(S, \preceq) \neq \emptyset\), and pick any \(\preceq\)-maximal element \(x\) in \(S\). Then, where \(y\) is a \(\preceq\)-maximum element in \(S\), we have \(y \succeq x\), but we cannot have \(y \succ x\) in view of the \(\preceq\)-maximality of \(x\) in \(S\). It follows that \(x \sim y\), so, by transitivity of \(\preceq\), we find that \(x \preceq \omega\) for every \(\omega \in X\), that is, \(x\) is a \(\preceq\)-maximum element in \(S\).

A nonempty subset \(S\) in a preordered set \((X, \preceq)\) may have several maxima, but of course, we have \(x \preceq y\) for any \(x, y \in \max(S, \preceq)\). In particular, if \((X, \preceq)\) is a poset, then there can be at most one \(\preceq\)-maximum element in \(S\). More generally, we have \(|\max(S, \preceq)| \leq 1\), provided that \(\preceq \cap (S \times S)\), which is a preorder on \(S\), is antisymmetric.

Let us now look at a few examples.

**Example 5.1.** Let \(X\) be a nonempty set. Then, a nonempty collection \(\mathcal{A}\) of subsets of \(X\) has a \(\supseteq\)-maximum if \(\bigcup \mathcal{A} \in \mathcal{A}\). Similarly, this collection has a \(\subseteq\)-minimum if \(\bigcap \mathcal{A} \in \mathcal{A}\). (In particular, the \(\supseteq\)-maximum of \(2^X\) is \(X\) and the \(\subseteq\)-minimum of \(2^X\) is \(\emptyset\).) By contrast, an element of \(\mathcal{A}\) is \(\supseteq\)-maximal iff it is not contained in any other element of \(\mathcal{A}\). For instance, for any nonempty proper subset \(S\) of \(X\), we have
\[
\max(\{S, X\setminus S\}, \supseteq) = \emptyset \subset \{S, X\setminus S\} = \text{MAX}(\{S, X\setminus S\}, \supseteq).
\]

**Example 5.2.** Let \(X\) be a set with \(|X| \geq 2\). Then,
\[
\text{MAX}(X, \Delta_X) = X = \text{MIN}(X, \Delta_X),
\]
while
\[
\max(X, \Delta_X) = \emptyset = \text{min}(X, \Delta_X).
\]

**Example 5.3.** Let \(S := \{(x, -x) : x \in \mathbb{R}\}\). Then,
\[
\text{MAX}(S, \geq) = S = \text{MIN}(S, \geq),
\]
while
\[
\max(S, \geq) = \emptyset = \text{min}(S, \geq).
\]
The following example shows that even when there is a unique maximal element in a poset, there may not be any maximum element.

**Example 5.4.** Consider the partial order $\succ$ on $\mathbb{N}$ defined by $i \succ i$ for each $i \in \mathbb{N}$, and $i \succ j$ iff $i > j > 1$. (This partial order agrees with the usual ordering of positive integers, except that it renders 1 as non-comparable to any other number in $\mathbb{N}$.) Then, there is a unique $\succ$-maximal element in $\mathbb{N}$, namely 1, while $\max(\mathbb{N}, \succ) = \emptyset$.

These examples demonstrate that, in a preordered set, there may be a considerable wedge between maximal and maximum elements. Yet, if the preorder under consideration is complete, this discrepancy disappears. That is, for any preordered set $(X, \succeq)$ and a nonempty subset $S$ of $X$, an element $x$ is $\succeq$-maximal in $S$ iff it is $\succeq$-maximum in $S$, provided that $\succeq$ is complete. (The same is true, of course, for minimal and minimum elements.)

A few comments on the existence of maximal and minimal elements are in order. Obviously, a poset need not have a maximal or a minimal element. (For instance, $\mathbb{R}$ has neither a $\geq$-maximal nor $\geq$-minimal element.) However, every finite preordered set is sure to possess extremal elements.

**Proposition 5.1.** Let $(X, \succeq)$ be a preordered set and $S$ a nonempty finite subset of $X$. Then,

$$\text{MAX}(S, \succeq) \neq \emptyset \neq \text{MIN}(S, \succeq).$$

**Proof.** The proof is by induction. The assertion is obviously true if $|S| = 1$. Then, take any positive integer $k$, and suppose our assertion holds for any subset of $X$ that contains $k$ many elements. Now take a subset $S$ of $X$ with $|S| = k + 1$. Pick any $x$ in $S$, and define $T := S \setminus \{x\}$. By the induction hypothesis, there exists a $\succeq$-maximal element, say, $y$, in $S$. Thus, if $x \succ y$ is false, $y$ is a $\succeq$-maximal element in $S$. If, on the other hand, $x \succ y$ holds, then $x$ is a $\succeq$-maximal element in $S$ (by $\succeq$-maximality of $y$ and transitivity of $\succ$). Thus, $\text{MAX}(S, \succeq) \neq \emptyset$. Applying this finding to the preordered set $(X, \succeq)$, we find also that $\text{MIN}(S, \succeq) \neq \emptyset$.

Evidently, finiteness of a poset is not enough for the existence of a maximum element (Example 5.2). But, if $\succeq$ is known to be complete, then an element is maximal in a set iff it is maximum in that set, and hence, Proposition 5.1 applies to yield:
Corollary 5.2. Let \((X, \succsim)\) be a preordered set and \(S\) a nonempty finite subset of \(X\). If \(\succsim\) is complete, then
\[
\max(S, \succsim) \neq \emptyset \neq \min(S, \succsim).
\]

Exercises
5.1. Give an example of a poset \((X, \succ)\) such that \(|X| \geq 2\) and there is an \(x \in X\) with \(\text{MAX}(S, \succ) = \{x\} = \text{MIN}(S, \succ)\).

5.2. A poset \((X, \succ)\) is said to be **well-founded** if every nonempty subset of \(X\) has a \(\succ\)-minimal element, and **well-ordered** if every nonempty subset of \(X\) has a \(\succ\)-minimum element. Give an example of a poset that is well-founded but not well-ordered.

5.3. An \(n \times n\) matrix \(B\) is said to be **bistochastic** if every term of this matrix is nonnegative, the sum of the terms of each row is 1, and the sum of the terms of each column is 1. We define the **Lorenz order** \(\geq_L\) as the binary relation on \(X := \{x \in \mathbb{R}_{+}^n : x_1 + \cdots + x_n = 1\}\) by \(x \geq_L y\) iff \(x = By\) for some \(n \times n\) bistochastic matrix. Prove that \(\geq_L\) is a partial order on \(X\), and compute \(\text{max}(X, \geq_L)\) and \(\text{min}(X, \geq_L)\).

5.4. Let \((X, \succsim)\) be a finite preordered set, and let \(\mathcal{P}\) stand for the collection of all completions of \(\succsim\). (Recall Exercise 3.7.) Prove:
\[
\text{MAX}(X, \succsim) = \bigcup \{\text{max}(X, \succ) : \succ \in \mathcal{P}\}.
\]

6. Suprema and Infima

The notion of supremum of a set of real numbers – that is, the smallest of all numbers greater than the numbers in that set – play an essential role in the order-theoretic construction of the real number system. This alone is enough of a motivation to explore this notion in the general context of posets. We will do this in this section. We shall later see that much of order theory takes its cue from the notions of supremum and infimum.

We begin with introducing the notions of upper and lower bounds for a subset of a poset.

**Notation.** In what follows, for any preordered set \((X, \succsim), x \in X\), and any subset \(S\) of \(X\), we write \(x \succsim S\) to mean \(x \succsim \omega\) for every \(\omega \in S\). The statement \(S \succsim x\) is similarly interpreted.

**Definition.** Let \((X, \succsim)\) be a preordered set. For any subset \(S\) of \(X\), an element \(x\) in \(X\) is said to be an \(\succsim\)-**upper bound** for \(S\) if \(x \succsim S\). Dually, \(x\) is said to be a \(\succ\)-**lower bound** for \(S\) if \(S \succ x\).
A maximum element of a set \( S \) in a preordered set is surely an upper bound for \( S \). The converse is false, of course. In fact, there may be numerous upper bounds for \( S \), and some of these bounds may be quite apart from \( S \). (For instance, every real number \( x \geq 1 \) is an \( \geq \)-upper bound for the interval \((0, 1)\), but 103 seems to have little to do with this set.) From the perspective of order theory, the larger these upper bounds are, the further apart they are from \( S \). Thus, the best proxy for a maximum element (of \( S \)) would be the smallest of all upper bounds for \( S \). (In the previous example, for instance, this proxy would be 1.) This prompts the following:

**Definition.** Let \((X, \succ)\) be a poset. For any subset \( S \) of \( X \), the \( \succ \)-supremum of \( S \) is the \( \succ \)-minimum element in the set of all \( \succ \)-upper bounds for \( S \). That is, an element \( x \) of \( X \) is the \( \succ \)-supremum of \( S \) if

- \( x \succ S \); and
- \( y \succ S \) implies \( y \succ x \) (for any \( y \in X \)).

(If there does not exist such an element \( x \) in \( X \), we then say that the \( \succ \)-supremum of \( S \) does not exist.) The \( \succ \)-inimum of \( S \) is defined dually (as the \( \succ \)-maximum of all \( \succ \)-upper bounds for \( S \)).

As there can be at most one maximum element in a poset, a supremum of a set (in a poset) is unique, provided that it exists. This is why we can talk about “the” supremum of a set, relative to a partial order. (The same goes also for “the” infimum of a set.)

**Notation.** For any poset \((X, \succ)\), and a subset \( S \) of \( X \), a natural notation for the \( \succ \)-supremum and \( \succ \)-inimum of \( S \) is \( \text{sup}(S, \succ) \) and \( \text{inf}(S, \succ) \). For historical reasons, however, it is common to use the notation

\[
\bigvee S \quad \text{and} \quad \bigwedge S
\]

for the \( \succ \)-supremum and \( \succ \)-inimum of \( S \), respectively. So long as one keeps in mind the ambient poset \((X, \succ)\) – in particular, notes that \( \bigvee S \) and \( \bigwedge S \), when they exist, are elements of \( X \), but not necessarily of \( S \) – using this notation would not cause a problem, and we shall do so here as well.

**Note.** For any poset \((X, \succ)\), and any subset \( S \) of \( X \),

\[
x \succ S \implies x \succ \bigvee S \quad \text{and} \quad S \succ x \implies \bigwedge S \succ x.
\]

Furthermore, \( \bigwedge S \) is the \( \preceq \)-supremum of \( S \).
Obviously, if a set $S$ (in a poset) has maximum (minimum) element, then that element is the supremum (infimum) of $S$. Here are some other examples.

**Example 6.1.** Let $(X, \succ)$ be a poset. Then, for any $x, y \in X$,

$$\vee \{x, y\} = x \quad \text{iff} \quad x \succ y \quad \text{iff} \quad \wedge \{x, y\} = y.$$ 

**Example 6.2.** Let $X$ be a set with $|X| \geq 2$. Then, relative to the poset $(X, \triangle_X)$, neither $\vee X$ nor $\wedge X$ exist.

**Example 6.3.** Let $(X, \succ)$ be a poset. Then, every element of $X$ is an $\succ$-upper bound for $\emptyset$. Consequently, $\vee \emptyset$ exists iff there is a $\succ$-minimum element in $X$, and when there is a $\succ$-minimum element in $X$, we have

$$\vee \emptyset = \text{ the } \succ\text{-minimum element in } X.$$ 

(Yes?) Similarly,

$$\wedge \emptyset = \text{ the } \succ\text{-maximum element in } X,$$

provided that a $\succ$-maximum element in $X$ exists. (If $\max(X, \succ) = \emptyset$, then $\wedge \emptyset$ does not exist.)

**Example 6.4.** Let $X$ be a nonempty set. Relative to the poset $(2^X, \supseteq)$, we have

$$\vee \mathcal{A} = \bigcup \mathcal{A} \quad \text{and} \quad \wedge \mathcal{A} = \bigcap \mathcal{A}$$

for any subset $\mathcal{A}$ of $2^X$ (provided that we adopt the convention that $\bigcup \emptyset = \emptyset$ and $\bigcap \emptyset = X$).

**Example 6.5.** Consider the poset $(\mathbb{R}^n, \geq)$, and let $S$ be a nonempty compact subset $S$ of $\mathbb{R}^n$. Then, we have

$$\vee S = (\max S_1, \ldots, \max S_n)$$

where $S_i := \{x_i \in \mathbb{R} : (x_i, x_{-i}) \in S \text{ for some } x_{-i} \in \mathbb{R}^{n-1}\}$, $i = 1, \ldots, n$, and similarly for $\wedge S$. Of course, $\vee S$ need not exist if $S$ is not compact. (For instance, neither $\vee \mathbb{R}^n$ nor $\wedge \mathbb{R}$ exist.)

**Example 6.6.** For each positive integer $m$, define $f_m \in C[0, 1]$ by $f_m(t) := t^m$, and let $S := \{f_1, f_2, \ldots\}$. Then, relative to the poset $(C[0, 1], \geq)$, we have $\vee S = f_1$, but $\wedge S$ does not exist. (Why?) On the other hand, $\wedge \{f_1, \ldots, f_k\} = f_k$ for any $k = 1, 2, \ldots$
Exercises

6.1. Let \((X, \geq)\) be a poset and \(S\) a nonempty subset of \(X\). Show that
\[
\bigvee S = \bigvee S^1,
\]
provided that either side of this equation exists.

6.2. Let \((X, \geq)\) be a poset. Consider the binary relation \(\geq\) on \(X\) defined by \(x \geq y\) iff for every \(\geq\)-directed (nonempty) subset \(D\) of \(X\) such that \(\bigvee D\) exists,
\[
\bigvee D \geq x \quad \text{implies} \quad \omega \geq y \quad \text{for some} \quad \omega \in D.
\]
Prove that \(\geq\) is antisymmetric and transitive, and give an example to show that it need not be reflexive. Also show that, if every nonempty subset of \(X\) has a \(\geq\)-maximal element, then \(\geq\) is a partial order.

7. Lattices

A poset \((X, \geq)\) is not sure to contain a smallest element that is greater than two of its elements, that is, \(\bigvee \{x, y\}\) may fail to exist for some \(x\) and \(y\) in \(X\). Example 6.2 provides a trivial instance of this possibility. Similarly, in \((\mathbb{R}^2, \geq)\), there is no \(\geq\)-supremum of the set \(\{(1,0), (0,1)\}\). Those posets in which such an irregularity does not occur are very important, and are given a special name in order theory.

Definition. Let \((X, \geq)\) be a poset. If \(\bigvee \{x, y\}\) and \(\bigwedge \{x, y\}\) exist for every \(x, y \in X\), we say that \((X, \geq)\) is a lattice. (If \(X\) is finite here, we say that \((X, \geq)\) is a finite lattice.) If \(\bigvee S\) and \(\bigwedge S\) exist for every subset \(S\) of \(X\) (including \(\emptyset\)), we then say that \((X, \geq)\) is a complete lattice.

Let us emphasize that a poset can fail to be a lattice for two reasons. First, a two-element set in the poset may not have an upper (or lower) bound. For instance, where \(X := \{(a, -a) : a \in \mathbb{R}\}\), the poset \((X, \geq)\) is not a lattice precisely for this reason. Second, even if every two-element set in the poset has an upper and a lower bound, it may be that there is no minimum (or maximum) element of the set of all upper (lower) bounds for at least one two-element set in that poset. For instance, where \(X\) is the union of \(\{(0,0), (0,1), (1,0)\}\) and \(\{(a, a) : a > 1\}\), the poset \((X, \geq)\) is not a lattice precisely for this reason. (Notice that every set in \((X, \geq)\) has at least one upper and one lower bound.)

Here are some other examples.
Example 7.1. Example 6.1 shows that every loset is a lattice. But, of course, a loset need not be a complete lattice. For instance, neither of the losets \([0, 1], \geq\) and \((\mathbb{R}, \geq)\) is a complete lattice.

Example 7.2. \((\mathbb{R}^n, \geq)\) is a lattice (but not a complete lattice). For every \(n\)-vectors \(x\) and \(y\), we have
\[
\vee\{x, y\} = (\max\{x_1, y_1\}, ..., \min\{x_n, y_n\}),
\]
and similarly for \(\wedge\{x, y\}\).

Example 7.3. \((C[0, 1], \geq)\) is a lattice. Indeed, for any two continuous real functions \(f\) and \(g\) on \([0, 1]\), \(\vee\{f, g\}\) is the continuous function that maps any \(x\) in \([0, 1]\) to \(\max\{f(x), g(x)\}\). Similarly, \(\wedge\{f, g\}\) corresponds to the pointwise minimum of \(f\) and \(g\). However, the poset \((C[0, 1], \geq)\) is a not a complete lattice, as we have seen in Example 6.6.

Example 7.4. By Example 6.4, \((2^X, \supseteq)\) is a complete lattice for any nonempty set \(X\).

It is useful to note that there is a maximum and minimum element in every complete lattice. Indeed, by Example 6.3, \(\wedge\emptyset\) is the \(\supseteq\)-maximum element in \(X\) and \(\vee\emptyset\) is the \(\supseteq\)-minimum element in \(X\), for any complete lattice \((X, \supseteq)\).

Note. A lattice need not have a maximum element – consider the loset \(([0, 1], \geq)\) – but this difficulty disappears in the case of lattices that possess a maximal element. Indeed, if \((X, \supseteq)\) is a lattice and \(x^*\) a \(\supseteq\)-maximal element in \(X\), then \(\vee\{x, x^*\}\) exists for any \(x\) in \(X\). But while \(\vee\{x, x^*\} \supseteq x^*\), the \(\supseteq\)-maximality of \(x^*\) in \(S\) implies \(x^* = \vee\{x, x^*\}\), and hence \(x^* \supseteq x\), for any \(x \in X\). Conclusion: A maximal element in a lattice is a maximum, and hence, a lattice can have at most one maximal element.

The following result provides two simple sufficient conditions for a poset to qualify as a complete lattice.

Proposition 7.1. A poset \((X, \supseteq)\) is a complete lattice if either of the following conditions hold:

\(\textbf{(a)}\) \((X, \supseteq)\) is a finite lattice;
\(\textbf{(b)}\) \(\vee S\) exists for every subset \(S\) of \(X\).
Proof. We prove part (a) here, leaving the proof of part (b) as an exercise. The argument is inductive. Suppose \((X, \geq)\) is a finite lattice, and pick any finite subset \(S\) of \(X\). If \(|S| = 1\), it is obvious that \(\bigvee S\) exists. Now let \(k\) be any positive integer, and as the induction hypothesis, suppose that the \(\geq\)-supremum of any subset of \(X\) with \(k\) elements exists. Now take any subset \(S\) of \(X\) with \(|S| = k + 1\). Pick any \(x\) in \(S\), and define \(T := S \setminus \{x\}\). By the induction hypothesis, \(\bigvee T\) exists. But then, as \((X, \geq)\) is a lattice, \(\bigvee \{\bigvee T, x\}\) exists. It is routine to check that \(\bigvee S = \bigvee \{\bigvee T, x\}\), so we conclude that \(\bigvee S\) exists. As the same argument can be made in terms of \(\leq\)-inma as well, we conclude that \(\bigvee S\) and \(\bigwedge S\) exist for every nonempty subset \(S\) of \(X\).

Corollary 7.2. Every finite lattice has a maximum and a minimum element.

Proof. By Proposition 7.1 every finite lattice is a complete lattice. But, as we have noted above, every complete lattice has a maximum and a minimum element.

It is worth noting that the “(complete) lattice” property is preserved under taking the product of two (or more) posets. That is:

Proposition 7.3. The product \((X \times Y, \geq)\) of two (complete) lattices \((X, \geq_X)\) and \((Y, \geq_Y)\) is a (complete) lattice.

Proof. This follows from the fact that, for any subset \(S\) of \(X \times Y\), we have
\[
\bigvee S = (\bigvee \{x : (x, y) \in S\}, \bigvee \{y : (x, y) \in S\}),
\]
and similarly for \(\bigwedge S\).

We now turn to those subsets of a lattice \((X, \geq)\) that not only inherit the partial order of \((X, \geq)\), but also the \(\geq\)-suprema and \(\geq\)-infima of finite sets from their mother lattice. More precisely:

Definition. Let \((X, \geq)\) be a lattice, and \(Y \subseteq X\). We say that \((Y, \geq)\) is a sublattice of \((X, \geq)\) if both \(\bigvee \{x, y\}\) and \(\bigwedge \{x, y\}\) belong to \(Y\) for every \(x, y \in Y\). (Here \(\bigvee \{x, y\}\) is the \(\geq\)-supremum of \(\{x, y\}\) in \(X\), and similarly for \(\bigwedge \{x, y\}\).) Similarly, we say that \((Y, \geq)\) is a subcomplete sublattice of \((X, \geq)\) if
\[
\bigvee S \in Y \quad \text{and} \quad \bigwedge S \in Y
\]
for every nonempty subset $S$ of $Y$. (Here $\bigvee S$ is the $\geq$-supremum of $S$ in $X$, and similarly for $\bigwedge S$.)

**Example 7.5.** For any integer $n \geq 2$,

$$Y := \left\{ x \in \mathbb{R}^n_+ : \sum_{i=1}^n x_i \leq 1 \right\}$$

is not a sublattice of $(\mathbb{R}^n, \geq)$. Indeed, the $i$th unit vector $e^i$ resides in $Y$, $i = 1, ..., n$, while

$$\bigvee \{e^1, ..., e^n\} = (1, ..., 1) \notin Y.$$ 

By contrast, $([0, 1]^n, \geq)$ is a (subcomplete) sublattice of $(\mathbb{R}^n, \geq)$.

**Example 7.6.** $(C^1[0, 1], \geq)$ is not a sublattice of the lattice $(C[0, 1], \geq)$, because the pointwise maximum of two continuously differentiable functions on $[0, 1]$ need not be differentiable. (In fact, $(C^1[0, 1], \geq)$ is not even a lattice.)

**Example 7.7.** Let $X$ be a nonempty set, and $\mathcal{A}$ a collection of subsets of $X$ that is closed under the operations of taking unions and intersections. Then, and only then, $(\mathcal{A}, \supseteq)$ is a subcomplete sublattice of $(2^X, \supseteq)$.

**Example 7.8.** Every order-interval in a lattice $(X, \succ)$ is a sublattice of $(X, \succ)$. For instance, for any $x$ in $X$, the ordered pair $(Y, \succ)$ is a sublattice of $(X, \succ)$, provided that $Y$ equals either $x^\downarrow$ or $x^\uparrow$. Similarly, $(Y, \succ)$ is a sublattice of $(X, \succ)$, where

$$Y = \{ \omega \in X : x \succ \omega \succ y \}$$

for some $x$ and $y$ in $X$ with $x \succ y$. It is also easy to see that $(Y, \succ)$ is a subcomplete sublattice of $(X, \succ)$ in any one of these cases, provided that $(X, \succ)$ is a complete lattice.

**Example 7.9.** Let $X$ be a nonempty compact subset of $\mathbb{R}^n$ such that $(X, \geq)$ is a sublattice of $\mathbb{R}^n$. Then, $X$ is a subcomplete sublattice of $(\mathbb{R}^n, \geq)$. To see this, take any nonempty subset $S$ of $X$. As $X$ is compact, $\text{cl}(S)$ is a compact subset of $\mathbb{R}^n$. Now, for each $i = 1, ..., n$, the map $x \mapsto x_i$ is continuous on $\mathbb{R}^n$ (and hence on $\text{cl}(S)$), so Weierstrass’ Theorem tells us that there exists an $n$-vector $x(i)$ in $\text{cl}(S)$ such that

the $i$th component of $x(i)$ $\geq$ the $i$th component of $x$ for all $x \in \text{cl}(S)$.
Then, because \((X, \geq)\) is a sublattice of \(\mathbb{R}^n\), \(\bigvee\{x(1), \ldots, x(n)\}\), that is, the \(\geq\)-supremum of \(\{x(1), \ldots, x(n)\}\) in \(\mathbb{R}^n\), belongs to \(X\). Let us call this \(n\)-vector \(x^*\). We wish to show that \(x^*\) is the \(\geq\)-supremum of \(S\) in \(\mathbb{R}^n\), that is, \(x^* = \bigvee S\).

As \(x^* \geq x(i)\), we have

\[ x_i^* \geq \text{the } i\text{th component of } x(i) \geq x_i \quad \text{for all } x \in \text{cl}(S), \]

\(i = 1, \ldots, n\). It follows that \(x^* \geq x\) for every \(x \in \text{cl}(S)\). In particular, \(x^*\) is an \(\geq\)-upper bound for \(S\). But, if \(y\) is another \(\geq\)-upper bound for \(S\) in \(\mathbb{R}^n\), it is readily checked that \(y \geq \text{cl}(S)\), and hence,

\[ y \geq \{x(1), \ldots, x(n)\}, \]

which implies \(y \geq x^*\). Thus, \(x^* = \bigvee S\), as we sought.

We have now seen that the \(\geq\)-supremum (in \(\mathbb{R}^n\)) of every nonempty subset of \(X\) belongs to \(X\). Applying this observation to the sublattice \((X, \leq)\), we find that \(X\) also contains the \(\geq\)-infimum (in \(\mathbb{R}^n\)) of each of its nonempty subsets. Conclusion: Every compact subset of \(\mathbb{R}^n\) which is a sublattice of \(\mathbb{R}^n\) is a subcomplete sublattice of \(\mathbb{R}^n\).

**Note.** The converse of this observation is also true, that is, if \((X, \geq)\) is a subcomplete sublattice of \(\mathbb{R}^n\), then \(X\) must be closed and bounded, and hence compact. We leave the proof of this fact as an exercise.

It is obvious that every (complete) sublattice of a (complete) lattice is a (complete) lattice (where, for instance, the \(\geq\)-supremum of \(\{x, y\}\) in \(Y\) is the \(\geq\)-supremum of \(\{x, y\}\) in \(X\)). However, it is important to note that the converse of this is false. That is, if \((X, \triangleright)\) and \((Y, \triangleright)\) are two lattices with \(Y \subseteq X\), there is no guarantee that \((Y, \triangleright)\) is a sublattice of \((X, \triangleright)\).

**Example 7.10.** Let \(Y\) consist of the vectors \((0, 0), (1, 0), (0, 1)\) and \((2, 2)\). Then \((Y, \geq)\) is a lattice, but it is not a sublattice of \((\mathbb{R}^2, \geq)\), because the \(\geq\)-supremum of \(\{(1, 0), (0, 1)\}\) in \(\mathbb{R}^2\) is \((1, 1)\). (The \(\geq\)-supremum of \(\{(1, 0), (0, 1)\}\) in \(Y\) is \((2, 2)\).)

A similar difficulty arises with respect to subcomplete sublattices. That is, a sublattice \((Y, \triangleright)\) of a complete lattice \((X, \triangleright)\) need not be a subcomplete sublattice of \((X, \triangleright)\), even though it is itself a complete lattice. (This is the reason why we do not use the term “complete sublattice” instead of the mouthful “subcomplete sublattice.”) We offer two illustrations of this possibility.
**Example 7.11.** Let \( Y := [0, 1) \). Then, \( (Y, \geq) \) is a complete lattice. It is, moreover, a sublattice of \( (\mathbb{R}, \geq) \). However, \( (Y, \geq) \) is not a subcomplete sublattice of \( (\mathbb{R}, \geq) \) for the \( \geq \)-supremum of \([0, 1)\) in \( \mathbb{R} \) is 1 which does not belong to \( Y \). (The \( \geq \)-supremum of \([0, 1)\) in \( Y \) is 2.)

**Example 7.12.** Let \( X \) be a topological space, and \( \mathcal{O}_X \) the collection of all open subsets of \( X \). Then, \( (\mathcal{O}_X, \supseteq) \) is a sublattice of \( (2^X, \supseteq) \). In fact, \( (\mathcal{O}_X, \supseteq) \) is a complete lattice as well. (Check!) However, as the intersection of infinitely many open sets need not be open, \( (\mathcal{O}_X, \supseteq) \) is not a subcomplete sublattice of \( (2^X, \supseteq) \) in general.

We conclude by noting that the “sublattice” property is preserved under taking the product of two (or more) lattices. That is:

**Proposition 7.4.** Let \( (Z, \geq_X) \) and \( (W, \geq_Y) \) be sublattices of the lattices \( (X, \geq_X) \) and \( (Y, \geq_Y) \), respectively. Then, the product of \( (Z, \geq_X) \) and \( (W, \geq_Y) \) is a sublattice of the product of \( (X, \geq_X) \) and \( (Y, \geq_Y) \).

We leave the proof as an (easy) exercise.

**Exercises**

7.1. Let \( X \) be the collection of all subsets \( S \) of \( \mathbb{N} \) such that \( \mathbb{N} \setminus S \) is finite. Is \( (X, \supseteq) \) a lattice? A complete lattice?

7.2. Let \( X \) be a nonempty set, and \( \mathcal{T}_X \) the collection of all topologies on \( X \). Show that \( (\mathcal{T}_X, \supseteq) \) is a complete lattice.

7.3. Let \( X \) be a nonempty set, and \( \mathcal{D}_X \) the collection of all metrics on \( X \). Is \( (\mathcal{D}_X, \supseteq) \) a lattice?

7.4. Let \( \succeq_1 \) be the Lorenz order on \( X := \{ x \in \mathbb{R}^n_+: x_1 + \cdots + x_n = 1 \} \) – recall Exercise 5.3. Is \( (X, \succeq_1) \) a lattice?

7.5. Let \( (X, \supseteq) \) be a preordered set. We say that a subset \( S \) of \( X \) is \( \supseteq \)-**increasing** if \( x^\uparrow \subseteq S \) for every \( x \in S \). (\( \supseteq \)-**decreasing** subsets of \( X \) are defined dually.) Let \( \mathcal{S} \) be the collection of all \( \supseteq \)-increasing subsets of \( X \). Show that \( (\mathcal{S}, \supseteq) \) is a complete lattice. Formulate a similar result for \( \supseteq \)-decreasing subsets of \( X \).

7.6. Let \( X \) be a nonempty set and \( X^\infty \) the collection of all sequences in \( X \). Define the \( \supseteq \) binary relation on \( X^\infty \) by \( (x_m) \supseteq (y_m) \) iff \( (y_m) \) is a subsequence of \( (x_m) \). Show that \( (X^\infty, \supseteq) \) is a poset which is not a lattice.

7.7. Is the linear sum of two (complete) lattices a (complete) lattice?
7.8. Let \((X, \succeq)\) be a lattice and \(S_X\) the collection of all sublattices of \((X, \succeq)\). Is \((S_X, \supseteq)\) a complete lattice?

7.9. Let \((X, \succeq)\) be a poset such that \(\bigvee S \in X\) for every nonempty subset \(S\) of \(X\) and \(\min(X, \succeq) \neq \emptyset\). Show that \((X, \succeq)\) is a complete lattice.

7.10. Use the previous exercise to show that \((O_X, \supseteq)\) is a complete lattice for any topological space \(X\). Also, give a formula for \(\bigwedge \mathcal{O}\) where \(\mathcal{O}\) is any collection of open subsets of \(X\).

7.11. Let \(X\) be a linear space and \(X'\) the collection of all linear subspaces of \(X\). Is \((X', \supseteq)\) a complete lattice?

7.12. (Order-Closure Operators) Let \((X, \succeq)\) be a poset. Let \(c\) be a \(\succeq\)-closure operator on \(X\), that is, \(c\) is a self-map on \(X\) such that (i) \(x \succeq y\) iff \(c(x) \succeq c(y)\); (ii) \(c(x) \succeq x\), and (iii) \(c(c(x)) = c(x)\), for every \(x, y \in X\). Show that 
\[c(X) = \{x \in X : x = c(x)\}.\]
Next, assume that \((X, \succeq)\) is a complete lattice, and prove that \((c(X), \succeq)\) is a complete lattice (but not necessarily a sublattice of \((c(X), \succeq))\). Also show that the \(\succeq\)-supremum of any subset \(S\) of \(c(X)\) in \(c(X)\) is \(c(\bigvee S)\) whereas the \(\succeq\)-infima of \(S\) in \(c(X)\) and in \(X\) coincide.

7.13. Use the previous exercise to prove that the following collections of sets are complete lattices with respect to the partial order \(\supseteq\):
   - (a) all convex subsets of a linear space;
   - (b) all linear subspaces of a linear space;
   - (c) all closed subsets of a topological space
   - (d) all increasing subsets of a poset
   - (e) all decreasing subsets of a poset.


7.15. Prove: If \((X, \succeq)\) is a subcomplete sublattice of \(\mathbb{R}^n\), then \(X\) must be compact.

8. Order-Preserving Functions

We now extend the notion of “monotonicity” of a real-to-real function to the context of maps between two preordered sets.

Definition. Let \((X, \succeq_X)\) and \((Y, \succeq_Y)\) be two preordered sets. A function \(f : X \to Y\) is said to be order-preserving if
\[x \succeq_X y \quad \text{implies} \quad f(x) \succeq_Y f(y),\]
and order-reversing if
\[x \succeq_X y \quad \text{implies} \quad f(y) \succeq_Y f(x),\]
for every $x$ and $y$ in $X$. (If $X = Y$ and $\preceq_X = \preceq_Y$ here, we say that $f$ is $\preceq$-preserving in the former case, and $\preceq$-reversing in the latter case.) Furthermore, when

$$x \preceq_X y \text{ if and only if } f(x) \preceq_Y f(y)$$

for every $x$ and $y$ in $X$, we say that $f$ is an order-embedding. (If there exists such a function, we say that $(X, \preceq_X)$ can be order-embedded in $(Y, \preceq_Y)$.) Finally, a bijective order-embedding is said to be an order-isomorphism. (If there exists such a function, $(X, \preceq_X)$ and $(Y, \preceq_Y)$ are said to be order-isomorphic.)

**Note.** The terminology is not entirely uniform in the literature. In particular, some authors use the term “isotonic” for “order-preserving.”

**Note.** Given any two preordered sets $(X, \preceq_X)$ to $(Y, \preceq_Y)$, an order-embedding $f$ is injective up to equivalence $\sim_X$, that is, $f(x) = f(y)$ implies $x \sim_X y$ for every $x, y \in X$. In particular, any order-embedding on a poset is an injection.

**Note.** The order-theoretic properties of order-isomorphic preordered sets are identical. Put differently, in the context of order theory, we can regard two order-isomorphic preordered sets as the “same,” for either one of these preordered sets can be obtained from the other by merely relabeling its elements.

The following examples provide some illustrations of these definitions.

**Example 8.1.** Let $(X, \preceq)$ be a preordered set. The identity function on $X$, and any constant function on $X$ (that is, a function that maps every element of $X$ to a fixed element of $X$) is an $\preceq$-preserving self-map on $X$. Such functions are called trivial $\preceq$-preserving self-maps on $X$. Among these maps, only the identity function is an order-embedding, unless we have $x \sim y$ for every $x$ and $y$ in $X$.

**Example 8.2.** A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is a $\preceq$-preserving self-map if it is increasing in each of its components. For instance, where $A$ is a nonnegative $n \times n$ matrix, the map $x \mapsto Ax$ is $\preceq$-preserving. If $A$ is invertible and $A^{-1}$ nonnegative, this map is an order-isomorphism.

**Example 8.3.** Consider the self-map $f$ on $\mathbb{Z}$ defined by $f(i) := i + 1$. Then, $f$ is an order-embedding (but not an order-isomorphism).
Example 8.4. The intervals $[0, 1]$ and $[0, 1)$ are not order-isomorphic. For, $[0, 1]$ has a $\geq$-maximum but $[0, 1)$ does not. Similarly, $\mathbb{Q}$ and $\mathbb{N}$ are not order-isomorphic. (How about $\mathbb{Q}$ and $\mathbb{Z}$?)

Example 8.5. Let $X$ be a topological space, and consider the self-map $f$ on $2^X$ defined by $f(S) := \text{cl}(S)$. Then, $f$ is a $\supseteq$-preserving self-map. This function is not an order-embedding. (After all, $f$ is not even injective. For instance, the closure of $\mathbb{Q}$ and $\mathbb{R}$ both equal $\mathbb{R}$.)

We next consider some basic properties of order-preserving maps.

Proposition 8.1. Let $(X, \preceq_X)$ and $(Y, \preceq_Y)$ be two preordered sets, and $f : X \to Y$ an order-preserving map. Then,

$$f(\max(S, \preceq_X)) \subseteq \max(f(S), \preceq_Y).$$

This is a trivial observation. However, it is important to note that the containment above may hold strictly. (Consider, for instance, the constant map $x \mapsto 1$ on $\mathbb{R}$.)

Proposition 8.2. Let $(X, \succeq_X)$ and $(Y, \succeq_Y)$ be two posets, and $f : X \to Y$ an order-preserving map. Then, for any subset $S$ of $X$,

$$f(\bigvee S) \succeq_Y \bigvee f(S)$$

(1)

provided that $\bigvee S$ and $\bigvee f(S)$ exist (in $X$ and $Y$, respectively). Similarly,

$$\bigwedge f(S) \succeq_Y f(\bigwedge S)$$

provided that $\bigwedge S$ and $\bigwedge f(S)$ exist (in $X$ and $Y$, respectively).

Proof. We only prove the first assertion, leaving the second as an exercise. Let $S$ be a subset of $X$ such that $\bigvee S$ and $\bigvee f(S)$ exist. As $\bigvee S$ is the $\succeq_X$-supremum of $S$ in $X$, it is an $\succeq_X$-upper bound for $S$, that is, $\bigvee S \succeq_X S$. As $f$ is order-preserving, therefore, we have $f(\bigvee S) \succeq_Y f(S)$, that is, $f(\bigvee S)$ is an $\succeq_Y$-upper bound for $f(S)$. But, by definition, $\bigvee f(S)$ is the $\succeq_Y$-minimum of such upper bounds, so (1) must be true.

Once again, the inequalities in this proposition may well hold strictly.
**Example 8.6.** Let $X$ be the subset of $\mathbb{R}^2$ that consists of $(1, 0)$, $(0, 1)$, $(1, 1)$ and $(2, 2)$. Consider the self-map $f$ on $X$ that maps $(1, 0)$ and $(0, 1)$ to $(1, 1)$, and $(1, 1)$ and $(2, 2)$ to $(2, 2)$. Then, $f$ is $\geq$-preserving, but for $S := \{(1, 0), (0, 1)\}$, we have

$$f(\bigvee S) = (2, 2) > (1, 1) = \bigvee f(S).$$

**Example 8.7.** Define the self-map on $2^\mathbb{R}$ by $f(S) := \text{cl}(S)$. Then, $f$ is a $\supseteq$-preserving self-map on $2^\mathbb{R}$, but where $Q := \{\{r\} : r \in Q\}$, we have

$$f(\bigvee Q) = f(\bigcup Q) = f(Q) = \text{cl}(Q) = \mathbb{R}$$

while

$$\bigvee f(Q) = \bigcup\{f(\{r\}) : r \in Q\} = Q.$$

However, it is obvious that all of the inequalities in Propositions 8.1 and 8.2 hold as equalities when the involved function is an order-isomorphism. That is:

**Proposition 8.3.** Let $(X, \preceq_X)$ and $(Y, \preceq_Y)$ be two posets, and $f : X \to Y$ an order-embedding. Then,

$$f(\text{MAX}(S, \preceq_X)) = \text{MAX}(f(S), \preceq_Y).$$

If $f$ is an order-isomorphism, we also have

$$f(\bigvee S) = \bigvee f(S) \quad \text{and} \quad f(\bigwedge S) = \bigwedge f(S),$$

whenever the involved suprema and infima exist.

The proof of this result is easy, and hence omitted.

**Exercises**

8.1. Prove that two finite lposets are order-isomorphic iff they contain the same number of elements.

8.2. Prove that $(\mathbb{Z}, \geq)$ and $(\mathbb{Q}, \geq)$ are not order-isomorphic. How about $(\mathbb{R}, \geq)$ and $(\mathbb{Q}, \geq)$?

8.3. Find two posets such that the first one can be order-embedded in the second one, and vice versa, while these posets are not order-isomorphic.

8.4. Let $(X, \succeq)$ be a poset and $\mathcal{S}$ the collection of all $\succeq$-increasing subsets of $X$. Show that $(\mathcal{S}, \supseteq)$ and $((\{0,1\}^X, \succeq)$ are order-isomorphic.

8.5. Let $(X, \preceq_X)$ and $(Y, \preceq_Y)$ be two posets and $f : X \to Y$ an order-preserving map. Prove: $f(S) \uparrow = f(S^\uparrow) \uparrow$ for any subset $S$ of $X$. 

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8.6. Let \((X, \succeq_X)\) and \((Y, \succeq_Y)\) be two posets. A function \(f : X \to Y\) is said to be \(\lor\)-preserving if
\[
f(\lor S) = \lor f(S)
\]
for every subset \(S\) of \(X\) such that \(\lor S\) exists. (The \(\land\)-preserving functions are dually defined.) Show that if \(f\) is either \(\lor\)-preserving or \(\land\)-preserving, then it must be order-preserving. Also, give an example of an order-preserving map that is neither \(\lor\)-preserving nor \(\land\)-preserving.

8.7. Let \((X, \succ)\) be a poset, and show that \(x \mapsto x^\downarrow\) is an order-embedding from this poset into \((2^X, \supseteq)\). (Thus: Every poset can be order-embedded in a complete lattice.)

8.8. Let \(\succ\) and \(\trianglerighteq\) be two partial orders on a nonempty set \(X\) such that a self-map on \(X\) is both \(\succ\)-preserving and \(\trianglerighteq\)-preserving iff it is trivial. Show that \(\succ \cap \trianglerighteq = \Delta_X\).

8.9. Let \((X, \succ)\) be a poset and \(Y\) a subset of \(X\) such that \((Y, \succeq)\) is a complete lattice. Show that there is an order-preserving map \(f : X \to Y\) such that \(f(y) = y\) for every \(y \in Y\).

8.10. Let \((X, \succ)\) be a complete lattice and \(f\) an \(\succ\)-preserving self-map on \(X\). Let \(Y := \{x \in X : x \succ f(x)\}\), and show that \((Y, \succeq)\) is a complete lattice. Also, give an example to show that \((Y, \succeq)\) need not be a sublattice of \((X, \succeq)\).

9. Galois Connections

We have noted above that two posets that are order-isomorphic are identical from the order-theoretic viewpoint. This is the strongest way in which two posets can be related to each other. Of course, there are weaker ways in which we can do this as well. For instance, two posets can be related to each other by means of a \(\lor\)-preserving and/or \(\land\)-preserving bijection, which means that insofar as the suprema and/or infima of their subsets are concerned, these two posets are identical. In this section, we shall outline a more subtle, but even more useful, manner in which two posets can be related to each other.

9.1 Definitions and Examples

The notion of Galois connections was introduced to order theory by Øystein Ore in 1944. The following is a somewhat more modern formulation of this concept.

Definition. Let \((X, \preceq_X)\) and \((Y, \preceq_Y)\) be two preordered sets, and \(f : X \to Y\) and \(F : Y \to X\) two functions. The ordered pair \((f, F)\) is said to be a Galois connection between \((X, \preceq_X)\) and \((Y, \preceq_Y)\) if
\[
y \preceq_Y f(x) \quad \text{iff} \quad F(y) \preceq_X x
\]
for every $x$ in $X$ and $y$ in $Y$. In this case, we say that $f$ is a **lower adjoint**, and $F$ is an **upper adjoint**, between $(X, \preceq_X)$ and $(Y, \preceq_Y)$.

**Note.** $(f, F)$ is a Galois connection between $(X, \preceq_X)$ and $(Y, \preceq_Y)$ iff $(f, F)$ is a Galois connection between $(Y, \preceq_Y)$ and $(X, \preceq_X)$. This is the **duality principle for Galois connections**.

**Note.** The terms “lower adjoint” and “upper adjoint” are hardly self-explanatory. Suffice it to say here that the roots of these terms come from category theory. (See Erné et al. (2006) for more on this.)

Here is our first example.

**Example 9.1.1.** Let $(X, \preceq_X)$ and $(Y, \preceq_Y)$ be two order-isomorphic preordered sets, and $f : X \to Y$ an order-isomorphism. Then, $(f, f^{-1})$ is a Galois connection between $(X, \preceq_X)$ and $(Y, \preceq_Y)$.

Therefore, being a lower adjoint is a less stringent requirement than being an order-isomorphism:

| order-isomorphism | $\implies$ | lower adjoint |

It is easy to see that the converse implication is false, that is, two preordered sets that are not order-isomorphic may well admit a Galois connection. In fact, even a loset (such as $(\mathbb{R}, \geq)$) can be Galois connected to a poset which is far from being a chain.

**Example 9.1.2.** Define the maps $f : \mathbb{R} \to \mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathbb{R}$ by

$$f(x) := (x, \ldots, x) \quad \text{and} \quad F(x) := \min\{x_1, \ldots, x_n\}.$$  

It is an easy exercise to check that $(f, F)$ is a Galois connection between $(\mathbb{R}, \geq)$ and $(\mathbb{R}^n, \succeq)$.

**Note.** A straightforward extension of the previous example allows us to look at the binary supremum and infimum operations on a lattice as lower and upper adjoints between that lattice and the product of two copies of that lattice, respectively. Indeed, if $(X, \geq)$ is a lattice, and the functions $f : X \to X \times X$ and $F : X \times X \to X$ are defined by $f(x) := (x, x)$ and $F(x, y) := \wedge\{x, y\}$, then $(f, F)$ is a Galois connection between $(X, \geq)$ and $(X \times X, \succeq_{X \times X})$. 

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As a matter of fact, instances of Galois connections abound across various branches of mathematics. As we shall see shortly, this allows one to use them to unify a large number of ostensibly different looking phenomena. We illustrate this next by means of a small selection of examples.

We begin with two examples from elementary set theory.

**Example 9.1.3.** Let $X$ and $Y$ be two nonempty sets, and $f : X \to Y$ and $F : Y \to X$ two functions. Then, $(f, F)$ is a Galois connection between $(X, \triangle_X)$ and $(Y, \triangle_Y)$ iff $F \circ f = \text{id}_X$ and $f \circ F = \text{id}_Y$. Put differently, $f$ is a lower adjoint between $(X, \triangle_X)$ and $(Y, \triangle_Y)$ iff $f$ is invertible and $f^{-1} = F$.

**Example 9.1.4.** Let $X$ and $Y$ be two nonempty sets, and $\varphi : X \to Y$ a function. Define the self-maps $f$ and $F$ on $2^X$ by

$$f(S) := \varphi(S) \quad \text{and} \quad F(T) := \varphi^{-1}(T).$$

(Here $\varphi^{-1}(T)$ stands for the inverse image of $T$ under $\varphi$, that is, it equals $\{x \in X : \varphi(x) \in T\}$.) It is easily checked that $(f, F)$ is a Galois connection between $(2^X, \supseteq)$ and itself.

Our next example is from topology.

**Example 9.1.5.** Let $X$ be a topological space, and recall that we denote the collection of all open subsets of $X$ by $\mathcal{O}_X$ and that of closed subsets of $X$ by $\mathcal{C}_X$. Define the maps $f : \mathcal{O}_X \to \mathcal{C}_X$ and $F : \mathcal{C}_X \to \mathcal{O}_X$ by

$$f(O) := \text{cl}(O) \quad \text{and} \quad F(C) := \text{int}(C).$$

Then, $(f, F)$ is a Galois connection between $(\mathcal{O}_X, \supseteq)$ and $(\mathcal{C}_X, \supseteq)$. Indeed, for any $C$ in $\mathcal{C}_X$ and $O$ in $\mathcal{O}_X$, it is obvious that $C \supseteq \text{cl}(O)$ implies that $C$ contains $O$. But then, since $O$ is open, it cannot contain a boundary point of $C$, that is, we have $\text{int}(C) \supseteq O$. Conversely, if $\text{int}(C) \supseteq O$, then

$$C = \text{cl}(C) \supseteq \text{cl}(\text{int}(C)) \supseteq \text{cl}(O).$$

It follows that $C \supseteq \text{cl}(O)$ iff $\text{int}(C) \supseteq O$, as we sought.

The following examples are from linear algebra and convex analysis, respectively.
Example 9.1.6. Let \((X, \langle \cdot, \cdot \rangle)\) be an inner product space, and recall that the \textbf{orthogonal complement} of a set \(S\) in this space is defined as

\[
S^\perp := \{ \omega \in X : \langle x, \omega \rangle = 0 \text{ for each } x \in S \}.
\]

It is easy to show that \(S^\perp\) is a closed linear subspace of \(X\) and we have \(S \subseteq S^\perp\), for every \(S \subseteq X\). It is also plain that \(S \subseteq T\) implies \(S^\perp \supseteq T^\perp\) for any two subsets of \(S\) and \(T\) of \(X\).

Now, let \(\mathcal{X}\) be the collection of all closed linear subspaces of \(X\), and define the maps \(f : 2^X \to \mathcal{X}\) and \(F : \mathcal{X} \to 2^X\) as

\[
f(S) := S^\perp \quad \text{and} \quad F(Y) := Y^\perp.
\]

Then, \((f, F)\) is a Galois connection between \((2^X, \supseteq)\) and \((\mathcal{X}, \subseteq)\). Indeed, if \(Y \subseteq f(S)\) for some subset \(S\) of \(X\) and a linear subspace \(Y\) of \(X\), then \(Y^\perp \supseteq f(S)^\perp\), and hence

\[
F(Y) = Y^\perp \supseteq S^\perp \supseteq S.
\]

That \(F(Y) \supseteq S\) implies \(Y \subseteq f(S)\) is similarly proved.

Example 9.1.7. For any nonempty subset \(S\) of \(\mathbb{R}^n\), we define the \textbf{polar} of \(S\) as

\[
S^\circ := \{ c \in \mathbb{R}^n : c_1 x_1 + \cdots + c_n x_n \leq 1 \text{ for each } x \in S \}.
\]

It is easy to check that \(S^\circ\) is a closed convex subset of \(\mathbb{R}^n\) that contains the origin, and we have \(S \subseteq S^{\circ \circ}\), for every \(S \subseteq \mathbb{R}^n\). It is also plain that \(S \subseteq T\) implies \(S^\circ \supseteq T^\circ\) for any two subsets of \(S\) and \(T\) of \(\mathbb{R}^n\).

Now, let \(\mathcal{X}\) be the collection of all closed and convex subsets of \(\mathbb{R}^n\) that contain the origin, and define the maps \(f : 2^{\mathbb{R}^n} \to \mathcal{X}\) and \(F : \mathcal{X} \to 2^{\mathbb{R}^n}\) as

\[
f(S) := S^\circ \quad \text{and} \quad F(Y) := Y^\circ.
\]

Then, \((f, F)\) is a Galois connection between \((2^{\mathbb{R}^n}, \supseteq)\) and \((\mathcal{X}, \subseteq)\). The argument is analogous to the one we gave in the previous example.

We conclude with an example from order theory. We shall later make good use of this example.

Example 9.1.8. Let \((X, \succ)\) be a poset, and define the self-maps \(f\) and \(F\) on \(2^X\) by

\[
f(S) := \{ x \in X : S \succ x \} \quad \text{and} \quad F(T) := \{ x \in X : x \succ T \}.
\]

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(That is, \( f(S) \) is the collection of all \( \geq \)-lower bounds for \( S \), and \( F(T) \) is that of all \( \geq \)-upper bounds for \( T \).) Then, \((f, F)\) is a Galois connection between \((2^X, \supseteq)\) and \((2^X, \subseteq)\). Indeed, for any subsets \( S \) and \( T \) of \( X \),

\[
S \geq y \text{ for every } y \in T \iff x \geq T \text{ for every } x \in S,
\]

that is, \( T \subseteq f(S) \) iff \( F(T) \supseteq S \).

### 9.2 Properties of Galois Connections

The following characterization of Galois connections are occasionally useful.

**Proposition 9.2.1.** Let \((X, \preceq_X)\) and \((Y, \preceq_Y)\) be two preordered sets, and \( f : X \to Y \) and \( F : Y \to X \) two functions. Then, \((f, F)\) is a Galois connection between \((X, \preceq_X)\) and \((Y, \preceq_Y)\) if, and only if, both \( f \) and \( F \) are order-preserving, and

\[
F(f(x)) \preceq_X x \quad \text{and} \quad y \preceq_Y f(F(y))
\]

for every \( x \) in \( X \) and \( y \) in \( Y \).

**Proof.** Let \((f, F)\) be a Galois connection between \((X, \preceq_X)\) and \((Y, \preceq_Y)\). Take any \( x \) in \( X \), and notice that \( f(x) \preceq_Y f(x) \) because \( \preceq_Y \) is reflexive. Then, as \((f, F)\) is a Galois connection, \( F(f(x)) \preceq_X x \). In turn, if \( x \) and \( z \) are two elements of \( X \) such that \( x \preceq_X z \), then \( F(f(x)) \preceq_X z \) by (2) and transitivity of \( \preceq_X \). We thus find \( f(x) \preceq_X f(z) \) because \((f, F)\) is a Galois connection. Conclusion: \( f \) is order-preserving and \( F(f(x)) \preceq_X x \) for every \( x \in X \). That \( F \) is order-preserving and the second part of (2) follow from this fact via the duality principle for Galois connections.

Conversely, assume that \( f \) and \( F \) are order-preserving and (2) is valid for every \( x \) in \( X \) and \( y \) in \( Y \). Then, for an arbitrarily fixed \((x, y) \in X \times Y \) with \( y \preceq_Y f(x) \), we have

\[
F(y) \preceq_X F(f(x)) \preceq_X x
\]

because \( F \) is order-preserving and (2) holds. By transitivity of \( \preceq_X \), therefore, \( F(y) \preceq_X x \). That \( F(y) \preceq_X x \) implies \( y \preceq_Y f(x) \) is similarly proved.

While being a left adjoint is weaker than being an order-isomorphism, Proposition 9.2.1 shows that it is stronger than being order-preserving:

\[
\text{order-isomorphism} \implies \text{lower adjoint} \implies \text{order-preserving map}
\]
The converse implication is false, that is, an order-preserving map from one preordered set to another need not be a lower adjoint between these preordered sets. For instance, the self-map we considered in Example 8.6 is order-preserving but it is not a lower adjoint. (It is not difficult to prove this from scratch, but we will shortly be able to see this as a triviality.)

In what follows, we focus on the slightly more special case of Galois connections between two posets. An important observation about such connections is the following:

**Proposition 9.2.2.** Let \((f, F)\) be a Galois connection between the posets \((X, \preceq_X)\) and \((Y, \preceq_Y)\). Then,

\[
f \circ F \circ f = f \quad \text{and} \quad F \circ f \circ F = F.
\]

**Proof.** Take any \(x\) in \(X\). By Proposition 9.2.1, we have \(F(f(x)) \preceq_X x\) and hence, as \(f\) is order-preserving,

\[
f(F(f(x))) \preceq_Y f(x).
\]

On the other hand, by reflexivity of \(\preceq_X\), we have \(F(f(x)) \preceq_X F(f(x))\), and hence, as \((f, F)\) is a Galois connection,

\[
f(x) \preceq_Y f(F(f(x))).
\]

As \(\preceq_Y\) is antisymmetric, therefore, \(f(F(f(x))) = f(x)\), establishing the first part of our assertion. The second part of this assertion follows from its first part and the duality principle for Galois connections.

Proposition 9.2.2 gives another perspective to the comparison of order-isomorphisms and lower adjoints. Obviously, if a map \(f\) is an order-isomorphism between two posets, then the inverse of \(f\) maps the image of any element in the domain back to that element. Put differently, in this case we have two maps, \(f\) and \(f^{-1}\), such that \(f\) transforms any one element of the domain to an element of the range and \(f^{-1}\) transforms that element back to where we started. Proposition 9.2.2 shows that something similar (but much weaker) holds when \(f\) is instead a lower adjoint. Apparently, in that case we have two maps, \(f\) and the upper adjoint \(F\), such that \(f\) transforms any one element \(x\) of the domain to an element \(y\) of the range and \(F\) transforms \(y\) back to another element \(z\) in the domain. It may be that \(z\) is distinct from \(x\), but it must be the case that \(f\) would transform \(z\) exactly where it maps \(x\), that is, we have \(f(x) = f(z)\).

As an immediate corollary of Proposition 9.2.2, we see now that while a lower adjoint may fail to be an order-isomorphism, it nevertheless acts as an order-isomorphism between a restriction of its domain and its (unrestricted) range. That is:
Corollary 9.2.3. Let \((f, F)\) be a Galois connection between the posets \((X, \succeq_X)\) and \((Y, \succeq_Y)\). Then, \(f|_{F(Y)}\) is an order-isomorphism between \((F(Y), \succeq_X)\) and \((f(X), \succeq_Y)\).

Here is a quick application of Proposition 9.2.2.

Example 9.2.1. For any open subset \(O\) of a topological space \(X\), we have

\[
\begin{align*}
\text{cl}(\text{int}(\text{cl}(O))) &= \text{cl}(O).
\end{align*}
\]

On the other hand, in the context of any inner product space \((X, \langle \cdot, \cdot \rangle)\), we have

\[
S^\perp = S^\perp
\]

for every subset \(S\) of \(X\). Similarly,

\[
S^\circ = S^\circ
\]

for every subset of \(\mathbb{R}^n\). These seemingly disparate results are unified under the theory of Galois connections. They are established at one stroke by applying Proposition 9.2.2 to Examples 9.1.5, 9.1.6 and 9.1.7.

By definition, a Galois connection determines a lower adjoint. We have so far said nothing about the converse of this fact (although we have implicitly acted as if the converse is true). Our next result shows that, in the case of posets, every lower adjoint is compatible with exactly one upper adjoint. Furthermore, it gives a precise formula for this upper adjoint.

Proposition 9.2.4. Let \((f, F)\) be a Galois connection between the preordered sets \((X, \preceq_X)\) and \((Y, \preceq_Y)\). Then,

\[
F(y) \in \max(\{x \in X : y \preceq_Y f(x)\}, \preceq_X)
\]

for every \(y\) in \(Y\).

Proof. Take any \(y\) in \(Y\), and define \(S := \{x \in X : y \preceq_Y f(x)\}\). As \((f, F)\) is a Galois connection between \((X, \preceq_X)\) and \((Y, \preceq_Y)\), we have \(F(y) \preceq_X x\) for every \(x \in S\), that is, \(F(y) \preceq_X S\). Moreover, by Proposition 9.2.1, \(y \preceq_Y f(F(y))\). We thus have \(F(y) \in S\). It follows that \(F(y)\) is a \(\succeq_X\)-maximum element of \(S\).

The following is an (almost) immediate consequence of Proposition 9.2.4.
Corollary 9.2.5. Let \((f, F)\) be a Galois connection between the posets \((X, \succeq_X)\) and \((Y, \succeq_Y)\). Then,

\[
F(y) = \text{the } \succeq_X -\text{maximum element of } \{x \in X : y \succeq_Y f(x)\}
\]

for every \(y\) in \(Y\).

9.3 \(\lor\)-Preservation

Our discussion so far does not make transparent why two Galois connected posets are in fact "closely related." This is obvious in the case of order-isomorphic posets, as any two such posets have identical order-theoretic properties. For instance, if one has a maximum and/or minimum element, so must the other. More generally, an order-isomorphism preserves suprema and infima (when they exist). We shall now show that Galois connected posets also share certain important order-theoretic properties. Indeed, it turns out that a lower adjoint surely preserves suprema (but not necessarily infima). This is perhaps the most important property of lower adjoints.

Proposition 9.3.1. Let \((f, F)\) be a Galois connection between the posets \((X, \succeq_X)\) and \((Y, \succeq_Y)\). Then, \(f\) is \(\lor\)-preserving, that is,

\[
f(\lor S) = \lor f(S)
\]

for every subset \(S\) of \(X\) such that \(\lor S\) exists. Similarly, \(F\) is \(\land\)-preserving.

Proof. Let \(S\) be a subset of \(X\) such that \(\lor S \in X\). By Proposition 9.2.1, \(f\) is order-preserving. As \(\lor S \succeq_X S\), therefore, \(f(\lor S) \succeq_Y f(S)\), that is, \(f(\lor S)\) is an \(\succeq_Y\)-upper bound for \(f(S)\). Suppose \(y\) is another \(\succeq_Y\)-upper bound for \(f(S)\), that is, \(y \succeq_Y f(S)\). Then, as \((f, F)\) is a Galois connection, \(F(y) \succeq_X S\). Therefore, \(F(y) \succeq_X \lor S\), and hence, \(y \succeq_Y f(\lor S)\) because \((f, F)\) is a Galois connection. Conclusion: \(f(\lor S)\) is the \(\succeq_Y\)-minimum of the set of all \(\succeq_Y\)-upper bounds for \(f(S)\). That is, \(f(\lor S) = \lor f(S)\), and our first assertion is established. Our second assertion follows from the first via the duality principle for Galois connections.

This result sharpens our understanding of where being a lower adjoint is located in the hierarchy of monotonicity properties. Apparently:

\[
\text{order-isomorphism} \implies \text{lower adjoint} \implies \lor\text{-preserving map} \implies \text{order-preserving map}
\]
Looking back at Example 8.6 now shows that the map \( f \) considered there is order-preserving, but as it does not preserve suprema, it is not a lower adjoint. (As we promised, this observation is now a triviality.) It is a bit harder to see that there are \( \bigvee \)-preserving maps that are not lower adjoints. We leave providing an example to this effect as an exercise.

While being \( \bigvee \)-preserving is in general less demanding than being a lower adjoint, it is remarkable that these two properties coincide for those maps whose domains are complete lattices. This is the main result of this section (whose proof is already suggested by Corollary 9.2.4).

**Theorem 9.3.2.** Let \((X, \succeq_X)\) be a complete lattice and \((Y, \succeq_Y)\) a poset. Then, a function \( f : X \to Y \) is \( \bigvee \)-preserving if, and only if, it is a lower adjoint between \((X, \succeq_X)\) and \((Y, \succeq_Y)\).

**Proof.** In view of Proposition 9.3.1, we only need to establish the “only if” part of this assertion. To this end, assume that \( f \) is \( \bigvee \)-preserving, and define \( F : Y \to X \) by

\[
F(y) = \bigvee \{ \omega \in X : y \succeq_Y f(\omega) \}.
\]

\( F \) is well-defined because \((X, \succeq_X)\) is a complete lattice. It is order-preserving because \( \succeq_Y \) is transitive, while \( f \) is order-preserving because every \( \bigvee \)-preserving map is. On the other hand, for any \( x \) in \( X \),

\[
F(f(x)) = \bigvee \{ \omega \in X : f(x) \succeq_Y f(\omega) \} \succeq_X x
\]

because \( \succeq_Y \) is reflexive. Finally, take an arbitrary \( y \) in \( Y \), and set \( S := \{ \omega \in X : y \succeq_Y f(\omega) \} \). Then, \( y \succeq_Y f(\omega) \) for each \( \omega \in S \), and hence, as \( f \) is \( \bigvee \)-preserving,

\[
y \succeq_Y \bigvee \{ f(\omega) \in X : \omega \in S \} = f(\bigvee S) = f(F(y)).
\]

We may now invoke Proposition 9.2.1 to conclude that \((f, F)\) is a Galois connection between \((X, \succeq_X)\) and \((Y, \succeq_Y)\).

Thanks to Theorem 9.3.2, to see if a given map from a complete lattice into a poset is \( \bigvee \)-preserving, we may check if that map is a lower adjoint between the involved posets. This is a useful strategy because there is a straightforward method for checking if a given map is a part of a Galois connection as its lower adjoint. After all, there is a unique upper adjoint for any lower adjoint. Indeed, combining Theorem 9.3.2 and Corollary 9.2.5 yields:
**Corollary 9.3.3.** Let \((X, \preceq_X)\) be a complete lattice and \((Y, \preceq_Y)\) a poset. Then, a function \(f : X \to Y\) is \(\bigvee\)-preserving if, and only if, \((f, F)\) is a Galois connection between \((X, \preceq_X)\) and \((Y, \preceq_Y)\), where

\[
F(y) = \text{the } \preceq_X \text{-maximum element of } \{x \in X : y \preceq_Y f(x)\}
\]

for every \(y\) in \(Y\).

We conclude with a minor application of Proposition 9.3.1.

**Example 9.3.1.** For any collection \(\mathcal{O}\) of open subsets of a topological space \(X\), we have

\[
\text{cl}(\bigcup \mathcal{O}) = \bigcup \{\text{cl}(O) : O \in \mathcal{O}\}.
\]

On the other hand, in the context of any inner product space \((X, \langle \cdot, \cdot \rangle)\), we have

\[
(\bigcup S)^\perp = \bigcap \{S^\perp : S \in S\}
\]

for every subset \(S\) of \(X\). Similarly,

\[
(\bigcup S)^\circ = \bigcap \{S^\circ : S \in S\}
\]

for every subset of \(\mathbb{R}^n\). We see here again the power of the theory of Galois connections. All these results are obtained readily by applying Proposition 9.3.1 to Examples 9.1.5, 9.1.6 and 9.1.7.

**9.4 Application: The Dedekind-MacNeille Completion**

Let \(X\) be a nonempty set. By definition, a \(\supseteq\)-closure operator \(c\) on \(2^X\) is a \(\supseteq\)-preserving self-map on \(2^X\) such that

\[
c(S) \supseteq S \quad \text{and} \quad c(c(S)) = c(S)
\]

for every subset \(S\) of \(X\). An important observation is that the range of such a self-map is a complete lattice (relative to \(\supseteq\)). Furthermore, for any subset \(S\) of \(c(X)\), we have

\[
\text{the } \supseteq\text{-supremum of } S \text{ in } c(X) = c(\bigcup S)
\]

and

\[
\text{the } \supseteq\text{-infimum of } S \text{ in } c(X) = \bigcap S.
\]

You were asked to establish even more general facts than these in Exercise 7.12.
A natural way of defining a \( \supseteq \)-closure operator on \( 2^X \) is by using a suitable Galois connection.

**Lemma 9.4.1.** Let \( X \) be a nonempty set and \( (f, F) \) a Galois connection between \( (2^X, \supseteq) \) and a poset \( (Y, \succcurlyeq) \). Then, \( F \circ f \) is a \( \supseteq \)-closure operator on \( 2^X \).

**Proof.** By Proposition 9.2.1, \( f \) and \( F \) are order-preserving, and hence \( F \circ f \) is a \( \supseteq \)-preserving self-map on \( 2^X \), while, by the same result, we have \( F \circ f(S) \supseteq S \) for every subset \( S \) of \( X \). Finally, by Proposition 9.2.2,

\[
F \circ f \circ F \circ f = F \circ (f \circ F \circ f) = F \circ f,
\]

and hence the lemma.

We shall now use this simple consequence of the theory of Galois connections to establish a famous result of order theory. Let \( (X, \succcurlyeq) \) be a poset. Define the self-maps \( f \) and \( F \) on \( 2^X \) by

\[
f(S) := \text{the collection of all } \succcurlyeq\text{-upper bounds for } S,
\]

and

\[
F(T) := \text{the collection of all } \succcurlyeq\text{-lower bounds for } T.
\]

In Example 9.1.8, we have observed that \( (f, F) \) is a Galois connection between \( (2^X, \supseteq) \) and \( (2^X, \subseteq) \), so by Lemma 9.4.1, \( F \circ f \) is a \( \supseteq \)-closure operator on \( 2^X \).

Next, consider the function \( \psi : X \to 2^X \) defined by

\[
\psi(x) := x^\perp.
\]

This map has a number of interesting properties. First, for every \( x \in X \), we have

\[
F(f(\{x\})) = F(x^\perp) = x^\perp,
\]

which means that \( \psi(X) \) is contained in the range of \( F \circ f \). Second, it is readily checked that

\[
x \succcurlyeq y \iff x^\perp \supseteq y^\perp
\]

for every \( x \) and \( y \) in \( X \). Conclusion: \( \psi \) is an order-embedding from \( X \) into \( F \circ f(2^X) \). Third, \( \psi \) is both \( \lor \)-preserving and \( \land \)-preserving. To see this, take any subset \( S \) of \( X \) with \( \lor S \in X \). We wish to show that \( \psi(\lor S) = \lor \psi(S) \). First, note that

\[
\psi(\lor S) = (\lor S)^\perp
\]

\[
= F(\{\lor S\})
\]

\[
= F \circ f \circ F(\{\lor S\})
\]

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by Proposition 9.2.2. Second, in view of (3), where $F \circ f$ acts as the order-closure operator $\tau$,

$$\forall \psi(S) = F \circ f(\bigcup \{x^\uparrow : x \in S\})$$

$$= F \circ f(\bigvee S^\uparrow)$$

$$= F \circ f \circ F(\{\bigvee S\}),$$

where the second equality follows from the fact that the $\succ$-upper bounds for $(\bigvee S)^\uparrow$ and $\bigcup \{x^\uparrow : x \in S\}$, that is, the images of these sets under $f$, coincide. We leave the proof for the $\wedge$-preservation property of $\psi$ (which is easier) as an exercise.

Finally, we note that $(F \circ f(2^X), \supseteq)$ is a complete lattice – it is called the Dedekind-MacNeille completion of $(X, \succ)$ – because the range of every $\supseteq$-closure operator on $2^X$ is.$^4$

We proved:

**Theorem 9.4.2.** Every poset can be order-embedded in a complete lattice by an order-embedding that is both $\bigvee$-preserving and $\bigwedge$-preserving.

Here is a simple illustration.

**Example 9.4.2.** The Dedekind-MacNeille completion of the loset $(\mathbb{N}, \succeq)$ “is” $(\{1, 2, ..., \infty\}, \geq)$. Indeed, for any set $S$ of positive integers, we have

$$F \circ f(S) = \begin{cases} 
\{1, ..., \max(S)\} & \text{if } \max(S) \neq \emptyset, \\
\{1\} & \text{if } S = \emptyset, \\
\mathbb{N} & \text{otherwise},
\end{cases}$$

where $f$ and $F$ are defined by (4) and (5), with $\succ$ being the usual order $\geq$ of $\mathbb{N}$. Therefore, the Dedekind-MacNeille completion of $(\mathbb{N}, \geq)$ is

$$(\{\{1\}, \{1, 2\}, ..., \{\mathbb{N}\}, \supseteq),$$

and obviously, this complete lattice is order-isomorphic to $(\{1, 2, ..., \infty\}, \geq)$.

We can always order-embed a given poset $(X, \succ)$ in a complete lattice. For instance, the map $x \mapsto x^\uparrow$ is an order-embedding from $(X, \succ)$ into $(2^X, \supseteq)$. What makes the

$^4$The Dedekind-MacNeille completion of an arbitrary poset was introduced by Holbrook MacNeille in 1937. As the construction of this completion is clearly inspired by how $\mathbb{Q}$ is extended to $\mathbb{R}$ by means of Dedekind cuts, Richard Dedekind’s name is also associated with it.
Dedekind-MacNeille completion of a poset special is that this completion departs from the original poset in a minimal way. For example, it follows from Theorem 9.4.2 that the Dedekind-MacNeille completion of a complete lattice is order-isomorphic to itself. More generally:

**Proposition 9.4.2.** Let \((X, \succsim)\) be a poset, and \((Y, \succeq)\) a complete lattice in which \((X, \succsim)\) can be order-embedded. Then, the Dedekind-MacNeille completion of \((X, \succsim)\) can also be order-embedded in \((Y, \succeq)\).

We leave the proof of this result as an exercise.

**Exercises**

9.1. Let \(X\) be a set and \(Y\) a nonempty subset of \(X\). Define the self-maps \(f\) and \(F\) on \(2^X\) by

\[ f(S) := S \cap Y \quad \text{and} \quad F(T) := T \cup (X \setminus Y). \]

Show that \((f, F)\) is a Galois connection between \((2^X, \supseteq)\) and itself.

9.2. Let \(X\) and \(Y\) be two topological spaces, and \(\varphi : X \to Y\) a continuous map. Define the self-maps \(f\) and \(F\) on \(C_X\) by

\[ f(S) := \text{cl}(\varphi(S)) \quad \text{and} \quad F(C) := \varphi^{-1}(C). \]

Show that \((f, F)\) is a Galois connection between \((C_X, \supseteq)\) and itself.

9.3. (Generalization of Example 9.1.6) Let \(X\) be a linear space, and denote the collection of all linear functionals on \(X\) by \(X^*\). Recall that the annihilator of a subset \(S\) of \(X\) is defined as

\[ S^\perp := \{L \in X^* : L(x) = 0 \text{ for each } x \in S\}. \]

In turn, for any linear subspace \(L\) of \(X^*\), we define

\[ S^\perp := \{x \in X : L(x) = 0 \text{ for each } L \in L\}. \]

Now let \(X\) stand for the collection of all linear subspaces of \(X^*\), and define the maps \(f : 2^X \to X\) and \(F : X \to 2^X\) by \(f(S) := S^\perp\) and \(F(L) := L^\perp\). Prove: \((f, F)\) is a Galois connection between \((2^X, \supseteq)\) and \((X, \subseteq)\).

9.4. Let \((X, \succsim)\) be a lattice and \(Y\) a subset of \(X\) such that for every \(x\) in \(X\) there is a subset \(S\) of \(Y\) with \(x = \bigvee S\). (Such a set \(Y\) is said to be \(\bigvee\)-dense in \(X\), and \(\bigwedge\)-dense subsets of \(X\) are dually defined.) Let \(S := \{y^\perp \cap Y : y \in Y\}\), and define \(f : S \to X\) by \(f(S) := \bigvee S\). Prove that \(f\) is a lower adjoint between \((S, \supseteq)\) and \((X, \succsim)\).

9.5. Prove: If \((f, F)\) is a Galois connection between the preordered sets \((X, \succeq_X)\) and \((Y, \succeq_Y)\), and \((g, G)\) a Galois connection between \((Y, \succeq_Y)\) and \((Z, \succeq_Z)\), then \((g \circ f, F \circ G)\) is a Galois connection between \((X, \succeq_X)\) and \((Z, \succeq_Z)\).
9.6. Let \((f, F)\) be a Galois connection between the posets \((X, \succeq_X)\) and \((Y, \succeq_Y)\). Prove that \(f\) is injective iff \(F\) is surjective.

9.7. Give an example of a \(\lor\)-preserving map between two posets which is not a lower adjoint between those posets.

9.8. Prove that the Dedekind-MacNeille completion of \((\mathbb{Q}, \geq)\) is \(([\infty, \infty], \geq)\).

9.9. Prove that the Dedekind-MacNeille completion of a poset is both \(\lor\)-dense and \(\land\)-dense in \(X\).


10. Order-Preserving Correspondences

10.1 Induced Set-Orderings

There are various interesting ways in which we can come up with a way of ordering the nonempty subsets of a given set by using a partial order defined on that set. We shall consider two ways of doing this here.

**Definition.** Let \((X, \succeq)\) be a preordered set. The **weak set-ordering** induced by \(\succeq\) is the binary relation \(\succeq^w\) on \(2^X \setminus \{\emptyset\}\) such that \(A \succeq^w B\) iff for every \((x, y) \in A \times B\),

\[
x \succeq w \text{ for some } w \in B \quad \text{and} \quad z \succeq y \text{ for some } z \in A.
\]

Given any preordered set \((X, \succeq)\), the weak set-ordering induced by \(\succeq\) is loyal to this preorder in the sense that it agrees with \(\succeq\) on the ranking of singleton sets, that is,

\[
\{x\} \succeq^w \{y\} \quad \text{iff} \quad x \succeq y
\]

for every \(x, y \in X\). Furthermore, the weak set-ordering is itself a preorder.

**Proposition 10.1.1.** If \((X, \succeq)\) is a preordered set, then \((2^X \setminus \{\emptyset\}, \succeq^w)\) is a preordered set.

We leave the proof of this fact as an exercise, but note that the antisymmetry of \(\succeq\) would not guarantee that of \(\succeq^w\), that is, the weak set-ordering induced by a partial order need not be a partial order. (For instance, we have \([0, 1] \succeq^w \{0, 1\}\) and \(\{0, 1\} \succeq^w [0, 1]\) simultaneously.) Having said this, we should also note that two sets \(A\) and \(B\) that are
rendered “equivalent” by the weak set-ordering must have the same extremal elements. That is, as one can easily verify,

\[ A \succ^w B \succ^w A \quad \text{implies} \quad \text{MAX}(A, \succ) = \text{MAX}(B, \succ), \]

and similarly for \( \succ \)-minimal elements of \( A \) and \( B \). In the case of posets, the same also goes for suprema and infima. More generally:

**Proposition 10.1.2.** Let \((X, \succ)\) be a poset. Then, for any nonempty subsets \( A \) and \( B \),

\[ A \succ^w B \quad \text{implies} \quad \bigvee A \succ \bigvee B, \]

provided that \( \bigvee A \) and \( \bigvee B \) exist.

**Proof.** Assume \( A \succ^w B \), and consider any element \( y \) of \( B \). By definition of \( \succ^w \), there is an element \( z \) of \( A \) such that \( z \succ y \). It follows that \( \bigvee A \succ y \), that is, \( \bigvee A \) is an \( \succ \)-upper bound for \( B \), and hence, \( \bigvee A \succ \bigvee B \), as we sought.

When \((X, \succ)\) is a lattice, we have another useful way of inducing a set ordering on \( 2^X \setminus \{\emptyset\} \).

**Definition.** Let \((X, \succ)\) be a lattice. The **strong set-ordering** induced by \( \succ \) is the binary relation \( \succ^s \) on \( 2^X \setminus \{\emptyset\} \) such that \( A \succ^s B \) iff for every \((x, y) \in A \times B\),

\[ \bigvee \{x, y\} \in A \quad \text{and} \quad \bigwedge \{x, y\} \in B. \]

Given any lattice \((X, \succ)\), the strong set-ordering induced by \( \succ \) is loyal to the partial order \( \succ \) just like the weak set-ordering. That is,

\[ \{x\} \succ^s \{y\} \quad \text{iff} \quad x \succ y \]

for every \( x, y \in X \). Furthermore, the strong set-ordering is quite close to being a partial order.

**Proposition 10.1.3.** If \((X, \succ)\) is a lattice, then \( \succ^s \) is antisymmetric and transitive.

We leave the proof of this observation as an exercise, but note that, in general, \( \succ^s \) need not be reflexive. For instance, if \( S \) is the subset of \( \mathbb{R}^2 \) that consists of \((1, 0)\) and
(0, 1), then $S \succeq^w S$ is false. But, of course, this difficulty disappears when ranking the sublattices of the original lattice. That is:

**Corollary 10.1.4.** Let $(X, \supseteq)$ be a lattice, and $S$ the collection of all subsets $S$ of $X$ such that $(S, \supseteq)$ is a sublattice of $(X, \supseteq)$. Then, $(S, \supseteq)$ is a poset.

Finally, a comparison of the weak and strong set-orderings is in order. Given a lattice $(X, \supseteq)$, as one might expect, $\supseteq^w$ is stronger than $\supseteq^w$ in the sense that

$$A \supseteq^w B \text{ implies } A \supseteq^w B$$

for any nonempty subsets $A$ and $B$ of $X$. (Indeed, if $A \supseteq^w B$, then, for any $(x, y)$ in $A \times B$, we have $z := \bigvee \{x, y\} \in A$ and $w := \bigwedge \{x, y\} \in B$, while we obviously have $x \supseteq w$ and $z \supseteq y$.) The converse is false. That is, $A \supseteq^w B$ may hold while $A \supseteq^w B$ does not. For instance, within the context of $(\mathbb{R}^2, \geq)$, we have

$$\{(1, 0), (0, 1), (2, 2)\} \supseteq^w \{(1, 0), (0, 1)\},$$

while $\supseteq^w$ cannot be replaced here with $\supseteq^w$ because $\supseteq^w$-supremum of $\{(1, 0), (0, 1)\}$, that is, $(1, 1)$, does not belong to the former set.

We conclude by noting the following corollary of Proposition 10.1.2 and the fact that $\supseteq^w$ is stronger than $\supseteq^w$.

**Corollary 10.1.5.** Let $(X, \supseteq)$ be a lattice. Then, for any nonempty subsets $A$ and $B$,

$$A \supseteq^w B \text{ implies } \bigvee A \supseteq \bigvee B,$$

provided that $\bigvee A$ and $\bigvee B$ exist.

### 10.2 Order-Preserving Correspondences

By a **correspondence** $\Gamma$ from a nonempty set $X$ into another nonempty set $Y$, we mean a map from $X$ into $2^Y \setminus \{\emptyset\}$. Thus, for each $x \in X$, the image $\Gamma(x)$ of $x$ under $\Gamma$ is a nonempty subset of $Y$. We write

$$\Gamma : X \Rightarrow Y$$

to denote that $\Gamma$ is a correspondence from $X$ into $Y$. Just as in the case of functions, $X$ is called the **domain** of $\Gamma$, and $Y$ the **codomain** of $\Gamma$. For any $S \subseteq X$, we let

$$\Gamma(S) := \bigcup \{\Gamma(x) : x \in S\}.$$
The set $\Gamma(X)$ is called the range of $\Gamma$. If $\Gamma(X) \subseteq X$, we refer to $\Gamma$ as a self-correspondence on $X$.

Of course, every function $f : X \to Y$ can be viewed as a particular correspondence from $X$ into $Y$. Indeed, there is no difference between $f$ and the correspondence $\Gamma : X \Rightarrow Y$ defined by $\Gamma(x) := \{f(x)\}$ (other than the formalism of using $\{\cdot\}$ in denoting the images under $\Gamma$). Conversely, if $\Gamma$ is single-valued, that is, $|\Gamma(x)| = 1$ for every $x \in X$, then it can be thought of as a function mapping $X$ into $Y$. It is thus customary to identify the terms “single-valued correspondence” and “function.”

Note. For any nonempty set $X$, there is a tight connection between the binary relations on $X$ and self-correspondences on $X$. First, every reflexive binary relation $R$ on $X$ induces a self-correspondence $\Gamma_R$ on $X$ by

$$\Gamma_R(x) := \{y \in X : y R x\}.$$

Second, if $\Gamma$ is a self-correspondence on $X$, then

$$\{(x, y) \in X \times X : y \in \Gamma(x)\}$$

is a binary relation on $X$.

Note. The literature is not unified in the way it refers to correspondences. Some mathematicians call them multifunctions, some many-valued maps, and still others refer to them as set-valued maps. In the economics literature, the term correspondence is widely used.

The induced set orderings we discussed in the previous subsection allow us to extend the notion of “order-preservation” to the context of correspondences in a straightforward manner.

**Definition.** Let $(X, \preceq_X)$ and $(Y, \preceq_Y)$ be two preordered sets. A correspondence $\Gamma : X \Rightarrow Y$ is said to be weakly order-preserving if

$$x \preceq_X y \implies \Gamma(x) \preceq_Y^w \Gamma(y),$$

for every $x$ and $y$ in $X$. If $X = Y$ and $\preceq_X = \preceq_Y$ here, then we say that $\Gamma$ is weakly $\preceq$-preserving.

Similarly:

**Definition.** Let $(X, \preceq_X)$ be a preordered set and $(Y, \preceq_Y)$ a lattice. A correspondence $\Gamma : X \Rightarrow Y$ is said to be strongly order-preserving if

$$x \preceq_X y \implies \Gamma(x) \preceq_Y^s \Gamma(y),$$

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for every $x$ and $y$ in $X$. If $X = Y$ and $\succeq_X = \succ_Y$ here, then we say that $\Gamma$ is **strongly $\succeq$-preserving**.

From the comparison of the weak and strong set-orderings it follows that every strongly order-preserving correspondence (whose codomain is a lattice) is weakly order-preserving, and not conversely.

As a final order of business here, we shall talk about the order-preserving selections from a correspondence.

**Definition.** Let $(X, \succeq_X)$ and $(Y, \succ_Y)$ be two preordered sets, and $\Gamma : X \Rightarrow Y$ a correspondence. A **selection** from $\Gamma$ is a function $f : X \rightarrow Y$ such that $f(x) \in \Gamma(x)$ for every $x \in X$.

It is frequently useful to be able to find order-preserving selections from a given correspondence. The following two results provide some simple sufficient conditions for being able to do this.

**Proposition 10.2.1.** Let $(X, \succeq_X)$ be a preordered set, $(Y, \succ_Y)$ a poset, and $\Gamma : X \Rightarrow Y$ a weakly order-preserving correspondence such that $\max((x), \succ_Y) \neq \emptyset$ for every $x \in X$. Then, there exists an order-preserving selection from $\Gamma$.

**Proof.** We define $f : X \rightarrow Y$ by

$$f(x) := \text{the } \succ_Y \text{-maximum element of } \Gamma(x).$$

By Proposition 10.1.2, $f$ is an order-preserving selection from $\Gamma$.

As the strong set-ordering is stronger than the weak set-ordering, the following is an immediate consequence of Proposition 10.2.1.

**Corollary 10.2.2.** Let $(X, \succeq_X)$ be a preordered set and $(Y, \succ_Y)$ a lattice. Let $\Gamma : X \Rightarrow Y$ be a strongly order-preserving correspondence such that $(\Gamma(x), \succ_Y)$ is a sublattice of $(Y, \succ_Y)$ for every $x \in X$. Then, there exists an order-preserving selection from $\Gamma$. 
Exercises

10.1. Let $f$ be an increasing self-map on $\mathbb{R}$, and define $\Gamma : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\Gamma(x) := \{ \omega \in \mathbb{R} : f(\omega) = x \}.$$ 

Is $\Gamma$ weakly $\geq$-preserving? Strongly $\geq$-preserving?

10.2. Let $f$ be an increasing self-map on $\mathbb{R}^n$, and define $\Gamma : \mathbb{R} \rightarrow \mathbb{R}^n$ by

$$\Gamma(x) := \{ \omega \in \mathbb{R}^n : f(\omega) = x \}.$$ 

Is $\Gamma$ weakly $\geq$-preserving? Strongly $\geq$-preserving?

10.3. Let $(X, \succ)$ be a lattice with no $\succ$-maximum and $\succ$-minimum elements. Define the self-correspondences $\Gamma$ and $\Upsilon$ by $\Gamma(x) := x^\downarrow$ and $\Upsilon(x) := x^\uparrow$, respectively. Are these correspondences weakly $\succ$-preserving? Strongly $\succ$-preserving?

10.4. Let $(X, \succ)$ be a lattice, and $Y$ and $Z$ two subsets of $X$ such that $(Y, \succ)$ and $(Z, \succ)$ are sublattices of $(X, \succ)$. Assume that

$$Z \cap y^\downarrow \cap x^\downarrow \succ Y \cap y^\downarrow \cap x^\downarrow$$

for every $x$ and $y$ in $X$ such that both of these sets are nonempty. Show that $Z \succ Y$.

10.5. Let $(X, \succ_X)$ be a finite poset and $(Y, \succ_Y)$ a lattice. Let $\Gamma : X \rightarrow Y$ be a strongly order-preserving correspondence such that $(\Gamma(x), \succ_Y)$ is a sublattice of $(Y, \succ_Y)$ for every $x \in X$. Prove: If $f$ is a selection from $\Gamma$, then

$$x \mapsto \bigvee \{ f(\omega) : x \succ_X \omega \in X \}$$

is an order-preserving selection from $\Gamma$.

10.6. Let $(X, \succ_X)$ be a loset and $(Y, \succ_Y)$ a lattice. Let $\Gamma : X \rightarrow Y$ be a strongly order-preserving correspondence such that $(\Gamma(x), \succ_Y)$ is a sublattice of $(Y, \succ_Y)$ for every $x \in X$. True or false: $(\Gamma(X), \succ_Y)$ is a sublattice of $(Y, \succ_Y)$.
References